Practical advanced analysis of steel frames considering lateral-torsional buckling

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Abstract

In this paper, the advanced analysis of 3D steel frames accounting for lateral-torsional buckling is presented. This analysis accounts for material and geometric nonlinearities of the structural system and its component members. Moreover, the problem associated with conventional advanced analysis, which do not consider lateral-torsional buckling, is overcome. An efficient way of assessing steel frame behavior including gradual yielding associated with residual stresses and flexure and second-order effect is presented. A case study shows that lateral-torsional buckling is a very crucial element to be considered in advanced analysis. The proposed analysis is shown to be an efficient and reliable tool ready to be implemented into design practice.

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1. Introduction

In current engineering practice, the interaction between a structural system and its component members is represented by the effective length factor. The effective length method generally provides a good design of framed structures. However, despite its popular use as a basis for design, the approach has major limitations. First, it does not give an accurate indication of the factor against failure because it does not consider the interaction of strength and stability between the member and structural system in a direct manner. It is well-recognized that the actual failure mode of the structural system often does not have any resemblance whatsoever to the elastic buckling mode of the structural system, which is the basis for the determination of the effective length factor, $K$. The second and perhaps the most serious limitation is probably the rationale of the current two-stage process in design: elastic analysis is used to determine the forces acting on each member of a structural system, whereas inelastic analysis is used to determine the strength of each member treated as an isolated member. There is no verification of the compatibility between the isolated member and the member as part of a frame. The individual member strength equations as specified in specifications are unconcerned with system compatibility. As a result, there is no explicit guarantee that all members will sustain their design loads under the geometric configuration imposed by the framework.

In order to overcome the difficulties of the conventional approach, advanced analysis should be directly performed. With the current available computing technology with advancement in computer hardware and software, it is feasible to employ second-order plastic-hinge analysis techniques for direct frame design. Most of the second-order plastic analyses can be categorized into one of the two types: (1) Plastic-zone; or (2) Plastic-hinge based on the degree of refinements used to represent yielding. The plastic-zone method uses the highest refinements while the elastic–plastic hinge method allows for significant simplifications. The typical load–displacements of the plastic analyses are illustrated in Fig. 1. One of the second-order plastic-hinge analyses called the “plastic-zone method” discretizes framed members into several finite elements.

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Also the cross-section of each finite element is further subdivided into many fibers [1–3]. Although the plastic-zone solution is known as an “exact solution”, it is yet to be used for practical design purposes. The applicability of the method is limited by its complexity requiring intensive computational time and cost. The real challenge in our endeavor is to make this type of analysis competitive in present construction engineering practices. A more simple and efficient way to represent inelasticity in frames is the second-order plastic-hinge method. Until now, several second-order plastic-hinge analyses for space structures were developed by Prakash and Powell [4], Liew et al. [5], and Kim et al. [6], among others. The benefit of the second-order plastic-hinge analyses is that they are efficient and sufficiently accurate for the assessment of strength and stability of structural systems and their component members. But these conventional 3D second-order plastic-hinge analyses cannot consider lateral-torsional buckling. Therefore, advanced analysis needs to consider that effect to enhance its capacity in predicting the behavior of structure accurately.

Early attempts to study the behavior of thin-walled structures include Bleich [7], Barsoum and Gallagher [8], Trahair and Kitipornchai [9], and Allen and Bulson [10]. But these studies focused on the numerical methods that permitted to treat only linear elastic lateral-torsional buckling. After that, geometric nonlinear elastic studies about thin-walled element considering lateral-torsional buckling were conducted by Bazant and El Nimeiri [11], Yang and McGuire [12], Chan and Kitipornchai [13], Conci andGattass [14], Chen and Blandford [15], and Kwak et al. [16]. Nonlinear inelastic analyses of lateral-torsional buckling were performed by Pi and Trahair [17], Gruttmann et al. [18], Battini and Pacoste [19]. However, the drawback of these methods is that they must use many elements to obtain the accurate result of complex structures. Recently, Kim et al. [20] developed a nonlinear analysis method that can consider lateral-torsional buckling effect. But this method cannot predict the real behavior of member accurately because it only considered lateral-torsional buckling strength using AISC-LRFD equation.

The purpose of this paper is to propose a practical advanced analysis method that can conduct nonlinear inelastic analysis considering lateral-torsional buckling. The stability functions and the refined plastic-hinge approach are reasonably applied into the beam-column formulation to take the advantage of computational efficiency. The local buckling effects are ignored. The shear, torsional, and warping effects on the cross-sectional plastic strength are not considered.

2. Stiffness matrix formulation

2.1. Virtual work equation

In the conventional beam–column approach, some coupling terms between the flexural and torsional displacements are excluded due to the simple expansion from 2D element to 3D element, so the lateral-torsional buckling cannot be predicted there. To overcome this obstacle, a virtual work equation including lateral torsional buckling effect is used. The linearized form of incremental virtual work equation of beam–column element having doubly symmetric cross-section with 14 degrees of freedom may be expressed as [21]

\[
\frac{1}{2} \int_0^L \left[ EA \delta (u'^2) + EI_y \delta (v'^2) + EI_z \delta (w'^2) + E C_{w0} \delta (\theta'^2_x) + G J \delta (\theta'^2_y) \right] dx + \int_0^L \frac{F_x}{2} \delta (v'^2 + w'^2) dx + \int_0^L \frac{K}{2} \delta (\theta'^2_x) dx - \int_0^L F_y \delta (u') dx - \int_0^L F_z \delta (u') dx - \int_0^L M_y \delta (\theta_y') dx - \int_0^L M_z \delta (\theta_z') dx - \int_0^L \frac{M_z}{2} \delta (v'w') dx = \delta \{f\} - \{f\}
\]  

in which \( E \) is the modulus of elasticity; \( G \) is the shear modulus; \( A \) and \( L \) are the area and length of element; \( I_y \) and \( I_z \) are the moment of inertia with respect to \( y \) and \( z \) axes; \( C_{w0} \) is the warping constant; \( K = F_y b (I_y + I_z) / A \) is the Wagner coefficient; and \( \{f\} \) and \( \{u\} \) are element force and displacement vectors.
The tangent stiffness matrix of a beam–column element can be derived from Eq. (1) as

\[ [K] = [K_e] + [K_n] \]  
(2)

where

\[ [K_e] = EA [K_{11}^e] + EI_y [K_{22}^e] + EI_z [K_{33}^e] + \frac{M_{xA} + M_{xB}}{L} ([K_{10}^e] + [K_{01}^e]) \]

\[ - \frac{M_{zA} + M_{zB}}{L} ([K_{10}^e] + [K_{01}^e]), \]  
(3a)

\[ [K_n] = EC_o [K_{11}^n] + GJ [K_{10}^n] \]

\[ + \frac{K_{x}}{2} ([K_{11}^n] + [K_{10}^n] + [K_{01}^n] + [K_{00}^n] - [K_{12}^n] - [K_{21}^n] - [K_{20}^n] - [K_{02}^n]) \]

\[ - \frac{M_{zA} + M_{zB}}{L} ([K_{10}^n] + [K_{01}^n] + [K_{00}^n]) \]

\[ + M_{zA}([K_{10}^n] + [K_{01}^n]) + M_{zA}([K_{00}^n] + [K_{01}^e] + [K_{00}^e]), \]  
(3b)

and

\[ [K_{11}^{ep}] = \int_0^L \frac{d^2 [N_{1}]^T}{dx^2} \frac{d^2 [N_{1}]}{dx^2} x^p dx \]  
(4)

in which the subscripts ‘g’ and ‘h’ represent u, v, w or \( \theta \); the superscripts ‘x’ and ‘zt’ represent the order of differentiation of the vector of interpolation functions \([N]\); and ‘p’ is the order of exponent to \( x \).

\[ [N_1] = \begin{bmatrix} 0 & N_1 & 0 & 0 & 0 & N_2 & 0 & 0 & 0 & N_3 & 0 & 0 & 0 & N_4 & 0 \end{bmatrix}, \]

\( (5a) \)

\[ [N_2] = \begin{bmatrix} 0 & 0 & N_1 & 0 & 0 & 0 & N_2 & 0 & 0 & 0 & N_3 & 0 & -N_4 & 0 & 0 \end{bmatrix}, \]

\( (5b) \)

\[ [N_3] = \begin{bmatrix} 0 & 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 \end{bmatrix}, \]

\( (5c) \)

In Eqs. (3a) and (3b), \( F_{xB} = P \) is the axial force; \( M_{xA}, M_{yB}, M_{zA}, M_{zB} \) are the end moments with respect to \( y \) and \( z \) axes.

### 2.2. Application of conventional beam–column method for axial and flexural stiffness terms in \([K_e]\)

Note that all terms in the stiffness matrix \([K_e]\) are the usual linear and nonlinear elastic terms for axial and flexural actions in the conventional beam–column method. Instead of using the interpolation functions \([N]\) in deriving \([K_e]\), the stability functions reported by Chen and Lui [22] are employed to accurately capture the second-order effects. Then a plastic-hinge model is used to account for inelastic behavior. The detailed procedure is presented in the following sections.

#### 2.2.1. Stability functions accounting for second-order effect

From Kim et al. [6], the incremental element force–displacement relationship for axial and flexural actions of three-dimensional beam–column element can be written as

\[
\begin{bmatrix}
P \\
M_{yA} \\
M_{zA} \\
M_{zB}
\end{bmatrix} = \begin{bmatrix}
\frac{F_A}{T} & 0 & 0 & 0 & 0 \\
0 & S_{1E} & S_{2E} & T & 0 \\
0 & S_{2E} & S_{1E} & T & 0 \\
0 & 0 & 0 & S_{3E} & S_{4E} & T
\end{bmatrix} \begin{bmatrix}
\delta \\
\theta_{yA} \\
\theta_{zA} \\
\theta_{zB}
\end{bmatrix},
\]

\( (7) \)

where \( \delta \) is the axial shortening, \( \theta_{yA}, \theta_{zB}, \theta_{zA}, \theta_{zB} \) are the joint rotations and \( S_1, S_2, S_3, S_4 \) are the stability functions with respect to \( y \) and \( z \) axes.

The stability functions in Eq. (7) may be written as

\[ S_1 = \begin{cases} 
\pi \sqrt{\rho_y} \sin(\pi \sqrt{\rho_y}) - \pi^2 \rho_y \cos(\pi \sqrt{\rho_y}) & \text{if } P < 0, \\
\frac{\pi^2 \rho_y \cosh(\pi \sqrt{\rho_y}) - \pi \sqrt{\rho_y} \sinh(\pi \sqrt{\rho_y})}{2 - 2 \cosh(\pi \sqrt{\rho_y})} & \text{if } P > 0, 
\end{cases} \]

\( (8a) \)

\[ S_2 = \begin{cases} 
\frac{\pi \sqrt{\rho_z} \sinh(\pi \sqrt{\rho_z})}{2 - 2 \cosh(\pi \sqrt{\rho_z})} & \text{if } P < 0, \\
\pi \sqrt{\rho_z} \sinh(\pi \sqrt{\rho_z}) - \pi^2 \rho_z & \text{if } P > 0, 
\end{cases} \]

\( (8b) \)

\[ S_3 = \begin{cases} 
\frac{\pi \sqrt{\rho_z} \sinh(\pi \sqrt{\rho_z})}{2 - 2 \cosh(\pi \sqrt{\rho_z})} & \text{if } P < 0, \\
\pi \sqrt{\rho_z} \sinh(\pi \sqrt{\rho_z}) - \pi^2 \rho_z & \text{if } P > 0, 
\end{cases} \]

\( (8c) \)
where \( \rho_y = P/(\pi^2EIy/L^2) \), \( \rho_z = P/(\pi^2EIz/L^2) \) and \( P \) is positive for tension.

The numerical solutions obtained from Eqs. (8a–8d) are indeterminate when the axial force is zero. To circumvent this problem and to avoid the use of different expressions for \( S_1, S_2, S_3, \) and \( S_4 \) for a different sign of axial forces, Lui and Chen [23] have proposed a set of expressions that make use of power-series expansions to approximate the stability functions. The power-series expressions have been shown to converge to a high degree of accuracy within the first ten terms of the series expansions. Alternatively, if the axial force in the member falls within the range \(-2.0 \leq \rho \leq 2.0\), the following simplified expressions may be used to closely approximate the stability functions [24]:

\[
S_1 = 4 + \frac{2\pi^2 \rho_y}{15} - \frac{(0.01 \rho_y + 0.543) \rho_y^2}{4 + \rho_y} - \frac{(0.004 \rho_y + 0.285) \rho_y^2}{8.183 + \rho_y},
\]

\[
S_2 = 2 - \frac{\pi^2 \rho_y}{30} + \frac{(0.01 \rho_y + 0.543) \rho_y^2}{4 + \rho_y} - \frac{(0.004 \rho_y + 0.285) \rho_y^2}{8.183 + \rho_y},
\]

\[
S_3 = 4 + \frac{2\pi^2 \rho_z}{15} - \frac{(0.01 \rho_z + 0.543) \rho_z^2}{4 + \rho_z} - \frac{(0.004 \rho_z + 0.285) \rho_z^2}{8.183 + \rho_z},
\]

\[
S_4 = 2 - \frac{\pi^2 \rho_z}{30} + \frac{(0.01 \rho_z + 0.543) \rho_z^2}{4 + \rho_z} - \frac{(0.004 \rho_z + 0.285) \rho_z^2}{8.183 + \rho_z}.
\]

Eqs. (9a)–(9d) are applicable for members in tension (positive \( P \)) and compression (negative \( P \)). For most practical applications, Eqs. (9a)–(9d) give an excellent correlation to the “exact” expressions given by Eqs. (8a)–(8d). However, for \( \rho \) other than the range of \(-2.0 \leq \rho \leq 2.0\), the exact stability functions (Eqs. (8a)–(8d)) should be used. The stability function approach uses only one element per member and maintains accuracy in the element stiffness terms and in the recovery of element end forces for all ranges of axial loads. In this formulation, all members are assumed to be adequately braced to prevent out-of-plane buckling, and their cross-sections are compact.

### 2.2.2. Model for gradual yielding

A significant number of meshes are necessary in order to trace the inelastic stress–strain relationship of each mesh [25]. The approach is widely used in the commercial softwares including ABAQUS, ANSYS, etc. Those softwares are good for research purposes but not for design. Since the purpose of this paper is to develop a practical tool for design, the plasticity is approximated by using the Column Research Council (CRC) tangent modulus and the parabolic function whose values are determined by member forces rather than by stress and strain relationship of each mesh. Although this approximation is used, the method predicts the system strength with a reasonable accuracy as shown in the verification study. This approach has been developed and used by many researchers [3,26–30].

#### 2.2.2.1. CRC tangent modulus model associated with residual stresses

The CRC tangent modulus concept is used to account for gradual yielding (due to residual stresses) along the members. From Chen and Lui [22], the CRC \( E_t \) is written as

\[
E_t = 1.0E \quad \text{for} \quad P \leq 0.5P_y,
\]

\[
E_t^4 \frac{P}{P_y} \left(1 - \frac{P}{P_y}\right) \quad \text{for} \quad P > 0.5P_y.
\]

#### 2.2.2.2. Parabolic function for gradual yielding due to flexure

The tangent modulus model is suitable for a member subjected to axial force, but not adequate for cases of both axial force and bending moment. A gradual stiffness degradation model for a plastic-hinge is required to represent the partial plastification effects associated with bending. When softening plastic-hinges are active at both ends of an element, the force–deflection equation may be expressed as [6]

\[
M = \begin{bmatrix}
M_{yA} & M_{yB} \\
M_{zA} & M_{zB}
\end{bmatrix}
= \begin{bmatrix}
0 & k_{iy} & k_{iy} & 0 & 0 \\
0 & k_{iy} & k_{iy} & 0 & 0 \\
0 & 0 & k_{iz} & k_{iz} & 0 \\
0 & 0 & k_{iz} & k_{iz} & 0
\end{bmatrix}
\begin{bmatrix}
\delta \\
\theta_{yA} \\
\theta_{yB} \\
\theta_{zA} \\
\theta_{zB}
\end{bmatrix},
\]

where

\[
k_{iy} = \eta_A \left( S_1 - \frac{S_2}{S_1}(1 - \eta_B) \right) \frac{E_tI_y}{L},
\]

\[
k_{iy} = \eta_A \eta_B S_2 \frac{E_tI_y}{L},
\]

\[
k_{iz} = \eta_A \left( S_3 - \frac{S_2}{S_3}(1 - \eta_B) \right) \frac{E_tI_z}{L},
\]

\[
k_{iz} = \eta_A \eta_B S_4 \frac{E_tI_z}{L},
\]

\[
k_{iz} = \eta_B \left( S_3 - \frac{S_2}{S_3}(1 - \eta_A) \right) \frac{E_tI_z}{L}.
\]
The terms $\eta_A$ and $\eta_B$ are scalar parameters that allow for gradual inelastic stiffness reduction of the element associated with plastification at ends $A$ and $B$. These terms are equal to 1.0 when the element is elastic, and zero when a plastic hinge is formed. The parameter $\eta$ is assumed to vary according to the parabolic function

$$\eta = 1.0 \quad \text{for} \quad x \leq 0.5,$$

$$\eta = 4x(1-x) \quad \text{for} \quad x > 0.5,$$

where $x$ is a force-state parameter that measures the magnitude of axial force and bending moment at the element end. The term $x$ is expressed in a modified version of Orbison full plastification surface of cross-section, as presented by McGuire et al. [31], as follows:

$$x = p^2 + m_z^2 + m_y^4 + 3.5p^2m_z^2 + 3.0p^6m_y^2 + 4.5m_y^2m_z^2,$$  

where $p = P/P_y$, $m_z = M_z/M_{pz}$ (strong-axis), $m_y = M_y/M_{py}$ (weak-axis).

Initial yielding is assumed to occur based on a yield surface that has the same shape as the full plastification surface and with the force-state parameter denoted as $x_0 = 0.5$. If the forces change so the force point moves inside or along the initial yield surface, the element is assumed to remain fully elastic with no stiffness reduction. If the force point moves beyond the initial yield surface, the element stiffness is reduced to account for the effect of plastification at the element end.

The element force–displacement relationship from Eq. (11) may be symbolically written as

$$\{f_e\} = [K_e]\{d_e\}$$

in which $\{f_e\}$ and $\{d_e\}$ are the end force and displacement vectors of a framed member expressed as

$$\{f_e\}^T = \{f_1, f_2, f_3, f_4, f_5\}^T,$$

$$\{d_e\}^T = \{d_1, d_2, d_3, d_4, d_5\}^T,$$

where

$$C_{ij} = -k_{ij}k_{ij} - k_{ij}^2 + k_{ij}A_{xy}GL\left(k_{ij} + k_{ij} + 2k_{ij} + A_{xy}GL\right)^{-1},$$

$$C_{ij} = -k_{ij}k_{ij} + k_{ij}^2 + k_{ij}A_{xy}GL\left(k_{ij} + k_{ij} + 2k_{ij} + A_{xy}GL\right)^{-1},$$

$$C_{ij} = k_{ij}k_{ij} + k_{ij}^2 + k_{ij}A_{xy}GL\left(k_{ij} + k_{ij} + 2k_{ij} + A_{xy}GL\right)^{-1},$$

$$C_{ijz} = k_{ij}k_{ijz} - k_{ijz}^2 + k_{ijz}A_{xy}GL\left(k_{ij} + k_{ij} + 2k_{ij} + A_{xy}GL\right)^{-1},$$

$$C_{ijz} = k_{ij}k_{ijz} + k_{ijz}^2 + k_{ijz}A_{xy}GL\left(k_{ij} + k_{ij} + 2k_{ij} + A_{xy}GL\right)^{-1},$$

$$C_{ijz} = -k_{ij}k_{ijz} + k_{ijz}^2 + k_{ijz}A_{xy}GL\left(k_{ij} + k_{ij} + 2k_{ij} + A_{xy}GL\right)^{-1}.$$

2.2.3. Stiffness matrix for axial and flexural terms

The sign convention for the positive directions of element end forces $\{f_e\}$ and displacements $\{d_e\}$ of a framed member in Eq. (15) is shown in Fig. 2. By comparing the two figures, the equilibrium and kinematic relationships can be expressed in symbolic form as

$$\{f_e\}^T = [T]^T\{d_e\},$$

$$\{d_e\}^T = [T]^T\{d_e\}^T.$$
$[T]_{5 \times 10}$ is a transformation matrix written as

$$
[T]_{5 \times 10} = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{2} & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 1 & 0 \\
0 & \frac{1}{2} & 0 & 0 & 1 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 1
\end{bmatrix}.
$$

(20)

Using the transformation matrix by equilibrium and kinematic relations, the force–displacement relationship of a framed member may be written as

$$
\{f_n\} = [K_e]\{d_L\},
$$

(21)

$[K_e]$ is the element stiffness matrix expressed as

$$
[K_e]_{10 \times 10} = [T]_{5 \times 10}^T [K_e]_{5 \times 5} [T]_{5 \times 10}.
$$

(22)

Eq. (22) can be partitioned as

$$
[K_e]_{10 \times 10} = \begin{bmatrix}
[K_{e1}] & [K_{e2}] \\
[K_{e2}]^T & [K_{e3}]
\end{bmatrix},
$$

(23)

where

$$
[K_{e1}] = \begin{bmatrix}
a & 0 & 0 & 0 & 0 \\
0 & b & 0 & 0 & c \\
0 & 0 & d & -e & 0 \\
0 & 0 & -e & g & 0 \\
0 & c & 0 & 0 & h
\end{bmatrix},
$$

(24a)

$$
[K_{e2}] = \begin{bmatrix}
-a & 0 & 0 & 0 & 0 \\
0 & -b & 0 & 0 & c \\
0 & 0 & -d & -e & 0 \\
0 & 0 & e & i & 0 \\
0 & -c & 0 & 0 & j
\end{bmatrix},
$$

(24b)

$$
[K_{e3}] = \begin{bmatrix}
a & 0 & 0 & 0 & 0 \\
0 & b & 0 & 0 & -c \\
0 & 0 & d & e & 0 \\
0 & 0 & e & m & 0 \\
0 & c & 0 & 0 & n
\end{bmatrix}
$$

(24c)

and

$$
a = \frac{E_i A}{L}, \quad b = \frac{C_{iiz} + 2 C_{ijz} + C_{ijz}}{L^2}, \quad c = \frac{C_{iiz} + C_{ijz}}{L},
$$

$$
d = \frac{C_{ijy} + 2 C_{ijy} + C_{ijy}}{L^2}, \quad e = \frac{C_{ijy} + C_{ijy}}{L},
$$

$$
g = C_{ijy}, \quad h = C_{ijz}, \quad i = C_{ijy}, \quad j = C_{ijy}, \quad m = C_{ijy}, \quad n = C_{ijz}.
$$

(25)

Eq. (22) is used to enforce no sidesway in the member. If the member is permitted to sway, an additional axial and shear force will be induced in the member. We can relate this additional axial and shear force due to a member sway to the member end displacements as

$$
\{f_s\} = [K_s]\{d_L\},
$$

(26)

where $\{f_s\}$ and $[K_s]$ are end force vector, end displacement vector, and the element stiffness matrix. They may be written as

$$
\{f_s\} = \begin{bmatrix} r_{s1} & r_{s2} & r_{s3} & r_{s4} & r_{s5} & r_{s6} & r_{s7} & r_{s8} & r_{s9} & r_{s10} \end{bmatrix}^T,
$$

(27)

$$
[K_s]_{10 \times 10} = \begin{bmatrix} [K_s] & -[K_s] \\
-[K_s]^T & [K_s]
\end{bmatrix},
$$

(28)

where

$$
[K_s] = \begin{bmatrix}
0 & a & -b & 0 & 0 \\
- \frac{a}{c} & 0 & 0 & 0 & 0 \\
-b & 0 & c & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

(29)

and

$$
a = \frac{M_{zA} + M_{zB}}{L^2}, \quad b = \frac{M_{sA} + M_{sB}}{L^2}, \quad c = \frac{P}{L}.
$$

(30)

It can be realized that the stiffness matrix $[K_{ns}]_{10 \times 10} = [K_{ns}]_{10 \times 10}^T [K_{ns}]_{10 \times 10}$ includes the terms representing axial and flexural actions of element in Fig. 2(b). In order to match the size of the general stiffness matrix considering lateral-torsional buckling effect shown in Fig. 3(b), the stiffness matrix $[K_{ns}]_{10 \times 10}$ is expanded into $[K_{ns}]_{14 \times 14}$ by adding the zero rows and columns corresponding to the remaining degrees of freedom aside from axial and flexural ones. Stiffness matrix $[K_{ns}]_{14 \times 14}$ is used instead of $[K_s]$ in Eq. (2) and can be expressed as follows:

$$
\{f_{ns}\} = [K_{ns}]\{d_L\}.
$$

(31)

The sign convention for the positive directions of $\{f_{ns}\}$ and $\{d_L\}$ are shown in Fig. 3.

2.3. Stiffness matrix $[K_{w}]$

The element force–displacement relationship from Eq. (1) for $[K_{w}]$ may be written as

$$
\{f_w\} = [K_w]\{d_L\},
$$

(32)

where $\{f_w\}$ is end force vector, and may be written as

$$
\{f_w\}^T = \begin{bmatrix} F_{xA} & F_{yA} & F_{zA} & M_{xA} & M_{yA} & M_{zA} & B_A & F_{xB} & F_{yB} & F_{zB} & M_{xB} & M_{yB} & M_{zB} & B_B \end{bmatrix}.
$$

(33)
By application of the interpolation function \([N]\), the stiffness matrix \([K_w]\) can be derived as
\[
[K_w]_{14\times14} = \begin{bmatrix} [K_w]_1 & [K_w]_2 \\ [K_w]_2^T & [K_w]_3 \end{bmatrix},
\]
where
\[
[K_w]_1 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a_1 & a & 0 & d \\
0 & 0 & 0 & a_2 & 0 & a & g \\
0 & a_1 & a_2 & k & a_3 & -a_4 & l \\
0 & a & 0 & a_3 & 0 & 0 & h \\
0 & 0 & a & -a_4 & 0 & 0 & i \\
0 & d & g & l & h & i & m
\end{bmatrix},
\]
\[
[K_w]_2 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a_7 & -a & 0 & -d \\
0 & 0 & 0 & a_8 & 0 & -a & -h \\
0 & -a_1 & 0 & a_9 & k & -a & 10 & l \\
0 & 0 & 0 & 0 & -a_3 & 0 & b & i \\
0 & 0 & 0 & a_4 & -b & 0 & -e \\
0 & -c & -g & -l & -j & f & n
\end{bmatrix},
\]
and
\[
[K_w]_3 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -a_7 & a & 0 & d \\
0 & 0 & 0 & -a_8 & 0 & a & h \\
0 & -a_7 & -a_8 & k & a_9 & -a_{10} & -l \\
0 & a & 0 & a_9 & 0 & 0 & -a_{11} \\
0 & 0 & a & -a_{10} & 0 & 0 & a_{12} \\
0 & d & h & l & -a_{11} & a_{12} & m
\end{bmatrix}.
\]

Fig. 3. General element end forces and displacements notation: (a) forces; (b) displacements.
\[ a_3 = \frac{M_{zA} + 2M_{zB}}{10}, \quad a_4 = \frac{M_{yA} + 2M_{yB}}{10}, \]
\[ a_5 = \frac{L(-3M_{zA} + M_{zB})}{30}, \quad a_6 = \frac{(3M_{yA} - M_{yB})}{30}, \]
\[ a_7 = \frac{-M_{yA} + 11M_{yB}}{10L}, \quad a_8 = \frac{-M_{zA} + 11M_{zB}}{10L}, \]
\[ a_9 = \frac{2M_{zA} + M_{zB}}{10}, \quad a_{10} = \frac{2M_{yA} + M_{yB}}{10}, \]
\[ a_{11} = \frac{L(M_{zA} - 3M_{zB})}{30}, \quad a_{12} = \frac{L(M_{yA} - 3M_{yB})}{30}. \]  

2.4. Induced moment matrix

Induced moment matrix is generated by finite rotations of semi-tangential torsional moment and quasi-tangential bending moment to yield the true equilibrium condition that satisfies the rigid body tests. Due to the lack of conjugateness between bending moments and displacement derivatives, the symmetry section will be restored when the element is connected to other elements. As a result, only the symmetric portion of the induced moment matrix [32] has to be assembled to form the structure stiffness matrix

\[ \{f_i\} = [K]_\text{local}\{d_L\} \]  

in which \( \{f_i\} \) and \([K]_\text{local}\) are end force vector, end displacement vector, and the element stiffness matrix considering induced moment

\[ [K]_{14\times14} = \begin{bmatrix} [K]_1 & 0 \\ 0 & [K]_2 \end{bmatrix}, \]  

(38)

where

\[
[K]_1 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -a & b & 0 & 0 & 0 & 0 \\
0 & 0 & b & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}, \]  

(39a)

\[
[K]_2 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}, \]  

(39b)

and

\[ a_3 = \frac{M_{zA}}{2}, \quad b = \frac{M_{yA}}{2}, \quad c = \frac{M_{zB}}{2}, \quad d = \frac{M_{yB}}{2}. \]  

2.5. Coordinate transformation

By combining Eqs. (31), (32), and (37), the general beam–column element force-displacement relationship may be written as

\[ \{f_L\} = [K]_\text{local}\{d_L\}, \]  

(41)

where

\[ \{f_L\} = \{f_n\} + \{f_w\} + \{f_i\}, \]  

(42)

\[ [K]_\text{local} = [K_n] + [K_w] + [K_i]. \]  

(43)

The structural stiffness matrix may be written as

\[ [K]_\text{global} = [\beta]^T[K]_\text{local}[\beta] \]  

(44)

The elements of \([\beta]\) matrix are the direction cosines of the force and displacement vectors, for the force and displacement vectors. For the three-dimensional framed element the \([\beta]\) matrix expands to

\[
[\beta] = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & [L] & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}, \]

(45)

where

\[
[L] = \begin{bmatrix}
l_x & m_x & n_x \\
l_y & m_y & n_y \\
l_z & m_z & n_z \\
\end{bmatrix}, \]

(46)

in which the symbols \(l, m, n\) with appropriate subscripts denote the direction cosine of the angle between the subscripted local axis and each of the global \(X, Y\) and \(Z\) axes.

3. Solution technique

Both the simple incremental and the incremental-iteration method are available for the analysis. In the simple incremental method, the applied load increment is automatically reduced to minimize the error when the change in the element stiffness parameter (\(\Delta\eta\)) exceeds a defined tolerance. In the incremental-iteration load approach, the structure is assumed to behave linearly at a particular cycle of calculation. Because of the linearization process, equilibrium may be violated and the external force may not always balance the internal force. This unbalance force must be reapplied to the structure. Then, the solution is obtained by iteration process until equilibrium is satisfied. As the stability limit point is approached in the analysis, convergence of the solution may be slow. To facilitate convergence, the applied load increment is automatically reduced. If the structure system is unstable,
the determinant of stiffness matrix becomes either zero or negative value and the program stops.

4. Numerical examples

Verifications are performed for the following four cases: Orbison’s six-story frame not governed by lateral-torsional buckling and three examples that may fail in lateral-torsional buckling. The first is to verify how well the proposed analysis predicts geometric and material non-linear behavior of frames. The second, third, and fourth are to show how the proposed analysis captures lateral-torsional buckling strength accurately. In addition, a case study is presented to show the lateral-torsional buckling effect in predicting strength of structure.

4.1. Orbison’s six-story space frame

Fig. 4 shows Orbison’s six-story space frame [30]. USFOS, a computer program for the progressive collapse analysis of steel offshore structures [33], and the result in Liew et al. [5] are used to verify the results from the proposed method.

The yield strength of all members is 250 MPa (36 ksi) and Young’s modulus is 206,850 MPa (30,000 ksi). The gravity loads are applied at the columns of every story and are equivalent to a uniform floor load of 9.6 kN/m² (200 psf). The wind loads are simulated by applying point loads of 53.376 kN (12 kips) in the Y-direction at every beam–column joint of the front elevation. The loads are proportionally applied until the frame collapses. The limit strength of the frame is reached at a load ratio of 0.996 in this study, as compared with load ratio of 0.995 from USFOS and 1.005 from Liew’s result (Table 1 and Fig. 5).

4.2. Simple beam under end moments

Fig. 6 shows a beam with rectangular cross-section under the action of equal end moments with both ends restrained against rotations about the X and Y axes. Material and section properties are: $E = 71,240$ N/mm², $G = 27,190$ N/mm², $A = 18$ mm², $I_y = 0.54$ mm⁴, $I_z = 1350$ mm⁴, $J = 2.16$ mm⁴, and span of beam $L = 200$ mm. The theoretical lateral-torsional buckling moment can be predicted as

$$M_{cr} = \frac{\pi \sqrt{EIGJ}}{L} = 1493.3 \text{ N-mm.} \quad (47)$$

In the geometrical nonlinear analysis, a disturbing torsional moment of 0.01 M is applied at midspan of the beam to initiate the displacement and twist. The limit strength of the beam is reached at a load ratio of 1.017 in this study. As shown Fig. 7, the lateral-torsional buckling load can be obtained accurately by using only two elements for the beam member. Also, Fig. 7 shows that the proposed method is a practical method in comparison with Liew’s result [5].

4.3. Simply supported right-angle frame under end moments

Fig. 8 shows a simply supported right-angle frame with rectangular section under equal end moments. The classical solution of this example was computed to be $M_{cr} = 622.2$ N-mm by Argyris et al. [34]. The proposed method predicts the critical lateral buckling moments of this example to be 671.2 N-mm ($= 1.079 M_{cr}$) and 642.3 N-mm ($= 1.032 M_{cr}$) corresponding to the use of...
one and two elements per member, while the numerical solutions obtained using 1, 5, 10, and 20 finite elements per member by Teh and Clarke [35] were 881.8, 632.4, 624.7, and 622.8 N mm, respectively.

### 4.4. Clamped right-angle frame under a tip load

Fig. 9 shows a clamped right-angle frame under a tip load $P$. A perturbation load $P_p = P/1000$ is applied perpendicularly to the plane of frame. The critical load is predicted to be $P_{cr} = 1.094$ N by the proposed method using only one element per member, while the computed critical loads of Teh and Clarke [35] using 1, 3, and 5 finite elements per member are 1.19, 1.10, and 1.09 N, respectively, and the critical loads of Argyris et al. [34] using 43 triangular shell and 10 beam finite elements per member are 1.163 and 1.088, respectively (Table 2).
4.5. Case study

A three-dimensional, one-bay, two-story frame was selected for the case study. Fig. 10 shows a sidesway uninhibited frame subjected to combined lateral and vertical loads. The stress–strain relationship was assumed to be elastic–perfectly plastic with a 250 MPa (36 ksi) yield stress and a 200,000 MPa (29,000 ksi) elastic modulus. W21 × 44 was used for all the members. Two analyses are compared in this case study: the proposed and the conventional advanced analyses.

In the proposed analysis, the structure collapsed by lateral-torsional buckling. The load-carrying capacity \( P_u \) in term of applied load of the structural system was evaluated to be 71.4 kN (16.05 kips). If lateral torsional buckling was ignored, the frame failed by flexural buckling, and the load-carrying capacity \( P_u \) of the structural system was calculated to be 104.2 kN (23.42 kips). As a result, the conventional advanced analysis over-estimates the load-carrying capacity of the frame by 46%. The load versus vertical displacement relationship at node “A” of the proposed and conventional advanced analyses are compared in Fig. 11.

5. Conclusions

Advanced analysis considering lateral-torsional buckling has been developed. The conclusions of this study are as follows:

(1) The proposed method appropriately traces the inelastic nonlinear behavior including lateral-torsional buckling.

(2) The proposed method can consider lateral-torsional buckling effect using a minimum number of elements. It shows that the proposed method is a more practical method than finite element method.

(3) Compared to LRFD, the proposed method provides more information on structural behavior through an advanced analysis considering lateral-torsional buckling of the entire system.

(4) The proposed analysis can be used in lieu of costly plastic zone analysis, and it can be a powerful tool for use in daily design.

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References


