Chapter 6. Energy and Numerical Methods

Introduction

Exact. Solution $\Rightarrow$ Differential Equation

Approx. Solution $\Rightarrow$ Energy and Numerical

- Energy : Elastic, Perfect ; function 이용
  - $\boxed{\text{Rayleigh-Ritz}}$, Galerkin Method
- Numerical ; Elastic, Inelastic, Perfect, Imperfect ; Segment
  - $\boxed{\text{Newmark}}$ $\boxed{\text{Numerical Integration}}$

Rayleigh-Ritz

① Boundary Condition을 만족하는 Deflection Shape 가정

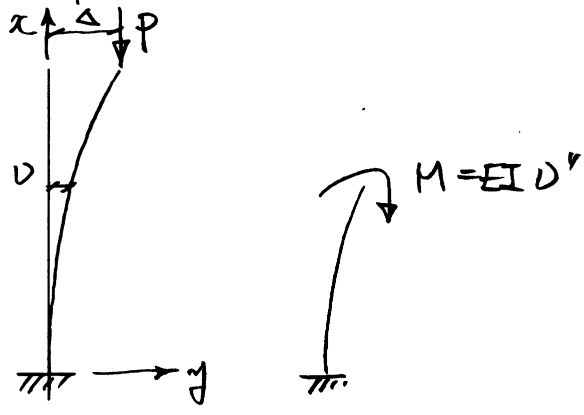
② Internal Energy (Strain Energy), $U(a)$

③ External Energy (Potential Energy), $V(P, a)$

④ Total Energy $\Pi = U + V$

⑤ $\dfrac{\partial \Pi}{\partial a} = 0 \Rightarrow P_{cr}$

Example 1. ; $v = ax^2$, curvature input in U



1) Assume Deflection

$$v = ax^2$$

2) Strain Energy.

$$U = U_a + U_b = \underbrace{\frac{1}{2}\int_0^L \frac{P^2}{EA}dx}_{\text{Neglect}} + \frac{1}{2}\int_0^L \frac{M^2}{EI}dx$$

axial shortening occurs before buckling

$$M = EI v''$$

$$U = U_b = \frac{1}{2}\int_0^L \frac{M^2}{EI}dx = \frac{1}{2}\int_0^L EI(v'')^2 dx$$

$$v'' = 2a$$

$$U = \frac{1}{2}\int_0^L EI\,(2a)^2 dx$$

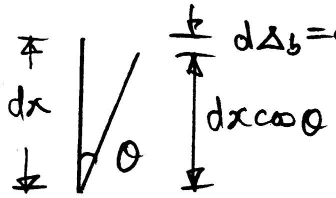
$$= 2EIa^2L$$

3) Potential Energy

$$V = V_a + V_b = -P\Delta_a - P\Delta_b$$

$$\Delta_a = \int_0^L \varepsilon_{xa}\, dx = \int_0^L \frac{du}{dx}\, dx$$

$$\Delta_b = \int_0^L d\Delta_b$$



$$d\Delta_b = dx\,(1-\cos\theta)$$

$$\cos\theta = 1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4$$

$$d\Delta_b = \frac{1}{2}\theta^2 dx = \frac{1}{2}\left(\frac{dv}{dx}\right)^2 dx$$

$$\Delta_b = \frac{1}{2}\int_0^L (v')^2 dx$$

$$V = -\frac{P}{2}\int_0^L (v')^2\, dx$$

$$v' = 2ax$$

$$V = -\frac{P}{2}\int_0^L 4a^2x^2\, dx$$

$$V = -\frac{2}{3}Pa^2L^3$$

4) Total Energy

$$\Pi = U + V = 2EIa^2L - \frac{2}{3}Pa^2L^3$$

5) P_{cr}

$$\frac{\partial \Pi}{\partial a} = \cancel{2}EI(\cancel{2a})L - \frac{\cancel{2}}{3}P(\cancel{2a})L^3 = 0$$

$$\underline{P_{cr} = \frac{3EI}{L^2}} \quad \text{vs. Exact } P_{cr} = \frac{\pi^2 EI}{4L^2} = 2.47\, EI/L^2$$

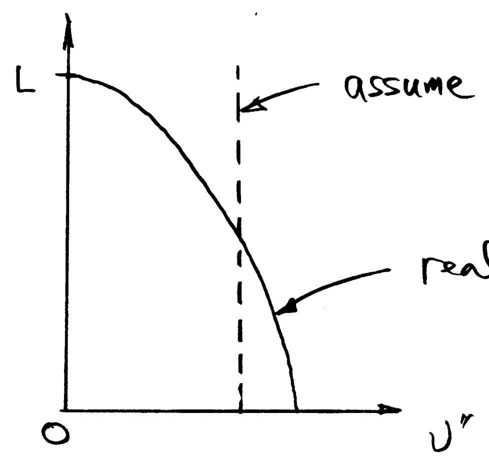
21% error

6) Discussion

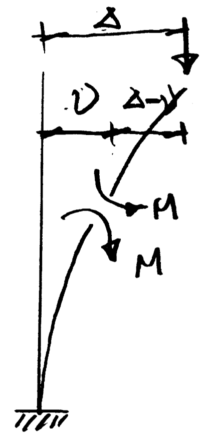
$v = ax^2$

$v'' = 2a$; curvature $\Rightarrow$ constant

Not real



Example 2: $v = ax^2$, displacement input in U



$v = ax^2$

$\Delta = aL^2$

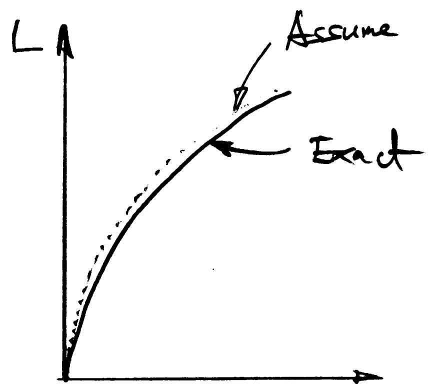
$M = P(\Delta - v) = Pa(L^2 - x^2)$

$$U = \frac{1}{2}\int_0^L \frac{M^2}{EI}dx = \frac{1}{2EI}\int_0^L \left[Pa(L^2-x^2)\right]^2 dx = \frac{8}{30}\frac{P^2a^2L^5}{EI}$$

$$\Pi = \frac{8}{30}\frac{P^2a^2L^5}{EI} - \frac{2}{3}Pa^2L^3$$

$$\frac{\partial \Pi}{\partial a} = \frac{8P^2aL^5}{15EI} - \frac{4}{3}PaL^3 = 0$$

$P_{cr} = 2.5\dfrac{EI}{L^2}$ vs. $P_{cr} = 2.47\, EI/L^2$ 1.2% error



Very Close

Because similar shape

※ Always upperbound in Rayleigh-Ritz

→ Displacement assumption ; mathematically constrained from nature shape

→ Stiffness increase effect

Example 3. $v = a_1 x^2 + a_2 x^3$

2 terms ; a_1, a_2

$v = a_1 x^2 + a_2 x^3 \longrightarrow$ can approach real displ.

$v' = 2a_1 x + 3a_2 x^2$

$v'' = 2a_1 + 6a_2 x$

$$\Pi = U + V$$

$$= \frac{1}{2}\int_0^L EI\,(v'')^2\,dx - \frac{P}{2}\int_0^L (v')^2\,dx$$

$$= \frac{1}{2}\int_0^L EI\,(2a_1 + 6a_2 x)^2\,dx - \frac{P}{2}\int_0^L (2a_1 x + 3a_2 x^2)^2\,dx$$

$\frac{\partial \Pi}{\partial a_1} = 0 \qquad 2EIL\,(2a_1 + 3a_2 L) - \frac{PL^3}{30}(40a_1 + 45a_2 L) = 0$

$\frac{\partial \Pi}{\partial a_2} = 0 \qquad 2EIL\,(3a_1 L + 6a_2 L^2) - \frac{PL^3}{30}(45a_1 L + 54a_2 L^2) = 0$

let $\lambda = \frac{PL^2}{EI}$

$$\begin{bmatrix} (24-8\lambda) & (36-9\lambda)L \\ (20-5\lambda) & (40-6\lambda)L \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

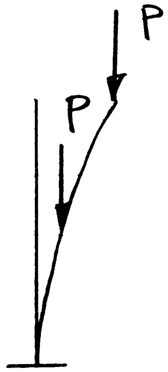
$$\det \begin{vmatrix} (24-8\lambda) & (36-9\lambda)L \\ (20-5\lambda) & (40-6\lambda)L \end{vmatrix} = 0.$$

$3\lambda^2 - 104\lambda + 240 = 0$

$\lambda = 2.49$

$P_{cr} = 2.49\,\frac{EI}{L^2}$ vs. $2.47\,\frac{EI}{L^2}$ 0.81% error

Example 4 , P_{cr}



$$v = a\left(1-\cos\frac{\pi x}{2L}\right)$$

$$v' = \frac{\pi}{2L}\sin\frac{\pi x}{2L}$$

$$v'' = \frac{\pi^2}{4L^2}\cos\frac{\pi x}{2L}$$

B.C. Check

$v(0)=0 \qquad v(L)=a$

$v'(0)=0$

$$U = \frac{1}{2}\int_0^L EI\,(v'')^2\,dx = \frac{EIa^2\pi^4}{64L^3}$$

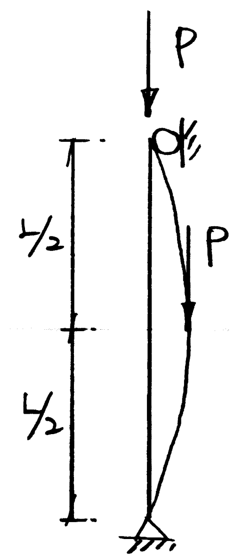
$$V = -\frac{2P}{2}\int_0^{L/2}(v')^2\,dx - \frac{P}{2}\int_{L/2}^{L}(v')^2\,dx = -\frac{Pa^2\pi}{32L}(3\pi-2)$$

$$\frac{\partial\Pi}{\partial a} = 0$$

$$P_{cr} = \frac{\pi^3}{2(3\pi-2)}\frac{EI}{L^2} = 2.09\frac{EI}{L^2} \quad \text{vs.} \quad 2.067\frac{EI}{L^2} \quad \text{(Exact)}$$

1.1% Error.

Example 5, P_{cr}



$$v = a \sin\frac{\pi x}{L}$$

$$v' = \frac{a\pi}{L}\cos\frac{\pi x}{L}$$

$$v'' = -\frac{a\pi^2}{L^2}\sin\frac{\pi x}{L}$$

Check B.C

$$\left.\begin{array}{l} v(0) = v(L) = 0 \\ v''(0) = v''(L) = 0 \end{array}\right) \text{o.k.}$$

$$U = \frac{1}{2}\int_0^L EI\,(v'')^2\,dx = \frac{EIa^2\pi^4}{2L^4}\int_0^L \left(\sin\frac{\pi x}{L}\right)^2 dx$$

$$= \frac{EIa^2\pi^4}{4L^3}$$

$$V = -\frac{2P}{2}\int_0^{L/2}(v')^2\,dx - \frac{P}{2}\int_{L/2}^L (v')^2\,dx$$

$$= -\frac{Pa^2\pi^2}{L^2}\int_0^{L/2}\left(\cos\frac{\pi x}{L}\right)^2 dx - \frac{2a^2\pi^2}{2L^2}\int_{L/2}^L\left(\cos\frac{\pi x}{L}\right)^2 dx$$

$$= -\frac{3Pa^2\pi^2}{8L}$$

$$\frac{\partial \Pi}{\partial a} = \frac{EIa\pi^4}{2L^3} - \frac{3Pa\pi^2}{4L} = 0$$

$$P_{cr} = \frac{2\pi^2 EI}{3L^2} = \frac{\pi^2 EI}{(1.2247\,L)^2}$$

H.W # 13

Problem 6.1 (b), (c)
6.3