# Chapter 4. Rigid Frames

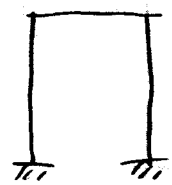
Rigid ; Connection is rigid

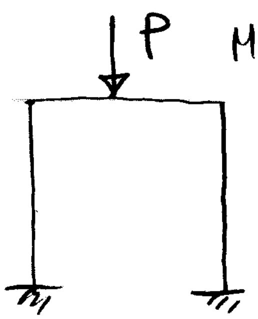
Frames ; structural system.

- Chap. 1, 2, 3 ; Isolated member
- Chap. 4 ; Whole system.

Interaction by adjacent members.

- beam → column
- col. → beam






Max. P ?

* Approach 1. ; Today

$P_{cr}$ (Elastic buckling load)
$P_p$ (Plastic collapse load)
> $P_{max}$ ($P_f$) ; failure load

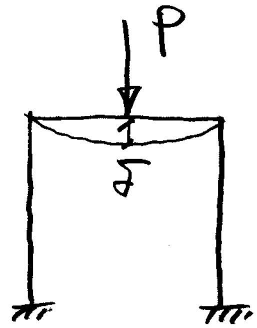
* Approach 2. ; Next week

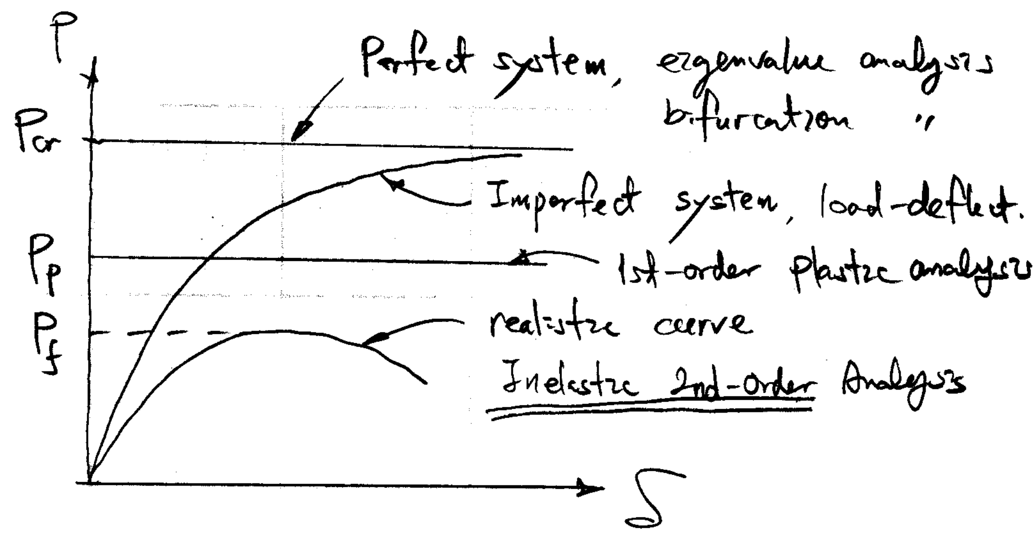
LRFD

Approach 3

Direct Nonlinear Inelastic Analysis : Advanced Analysis

## Approach 1 : Using $P_{cr}$, $P_p$



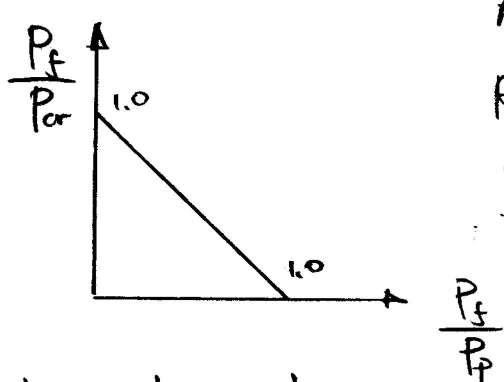




$$\frac{P_f}{P_{cr}} + \frac{P_f}{P_p} = 1.0$$ : Merchant - Rankine equation

Approximate equation

Reasonably accurate for design

Conservative




$$\frac{1}{P_{cr}} + \frac{1}{P_p} = \frac{1}{P_f}$$

## Elastic Buckling Load, $P_{cr}$

Methods (Eigenvalue Analysis)

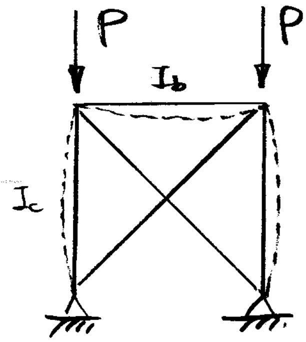
① Differential Eq.

② Slope-Deflection Eq. *

③ Matrix Stiffness Method *

### Lower and Upper Bound of K-factor

Braced




$P_{cr}$?

$K_{Lower} = 0.7$ $(I_b = \infty)$

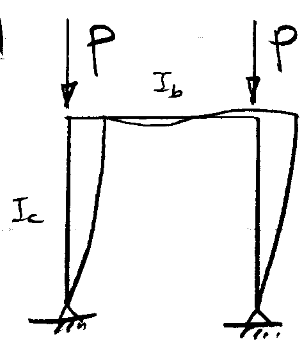
$K_{upper} = 1.0$ $(I_b = 0)$

$0.7 < K < 1.0$

$K = 0.875$ $(I_c = I_b)$

$$P_{cr} = \frac{\pi^2 EI}{(KL)^2}$$

Unbraced




$K_{Lower} = 2.0$ $(I_b = \infty)$

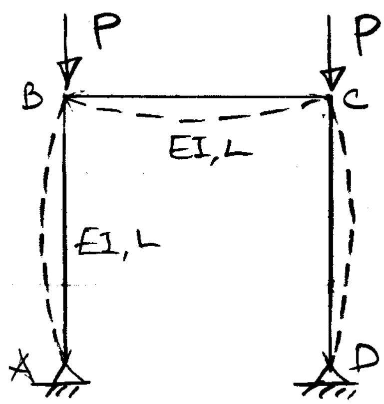
$K_{upper} = \infty$ $(I_b = 0)$

$2.0 < K < \infty$

$K = 2.33$ $(I_c = I_b)$

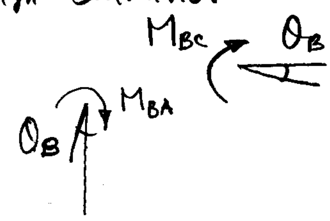
# Slope - Deflection Eq.

## 1) Braced Case




Symmetry Case

Sign Convention




Unknown ; $\theta_A$, $\theta_B$

Condition ; $M_{AB}=0$, $M_{BA}+M_{BC}=0$

Col.

$$M_{AB}=\frac{EI}{L}(S_{iic}\theta_A+S_{ijc}\theta_B)=0 \Rightarrow \theta_A=-\frac{S_{ijc}}{S_{iic}}\theta_B$$

$$M_{BA}=\frac{EI}{L}(S_{ijc}\theta_A+S_{iic}\theta_B)=\frac{EI}{L}\left(S_{iic}-\frac{S_{ijc}^2}{S_{iic}}\right)\theta_B$$

Beam.

$$M_{BC}=\frac{EI}{L}(S_{iib}\theta_B+S_{ijb}\theta_C)$$

$$\theta_C=-\theta_B$$

$$M_{BC}=\frac{EI}{L}(S_{iib}-S_{ijb})\theta_B = \frac{2EI}{L}\theta_B$$

Equil.

$$M_{BA}+M_{BC}=0$$

$$\frac{EI}{L}\left(S_{iic}-\frac{S_{ijc}^2}{S_{iic}}\right)\theta_B+\frac{2EI}{L}\theta_B=0$$

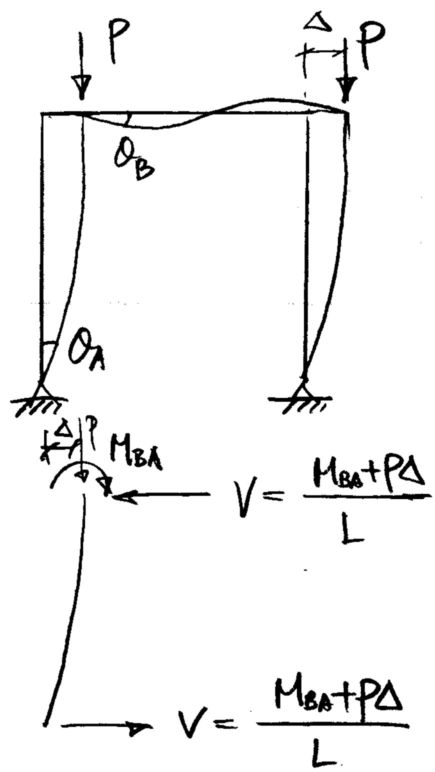
$$S_{iic}-\frac{S_{ijc}^2}{S_{iic}}+2=0$$

From Table 3.7, $kL=3.59$ $\quad \sqrt{\frac{P}{EI}}L=3.59$

$P_{cr}=12.9\frac{EI}{L^2} \rightarrow K=\sqrt{\frac{P_e}{P_{cr}}}=0.87^{\dagger}$

2) Unbraced Case

Antisymmetric




Unknown ; $\theta_A$, $\theta_B$, $\Delta$

Condition ; $M_A = 0$

$\Sigma M_B = 0 \quad M_{BA} + M_{BC} = 0$

$\Sigma V = 0 \quad \dfrac{M_{BA} + P\Delta}{L} = 0$

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{Bmatrix} \theta_A \\ \theta_B \\ \Delta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$\det | \quad | = 0$

$kL = 1.35$

$P_{cr} = 1.82 \dfrac{EI}{L^2}$

$K = 2.33$

or

$$s_{iic} - \frac{s_{ijc}^2}{s_{iic}} + 2 = 0 \tag{4.3.9}$$

By trial and error and using Table 3.7, the value of $kL$ that satisfies Eq. (4.3.9) is found to be

$$kL = (\sqrt{P/EI})L = 3.59 \tag{4.3.10}$$

from which the critical load

$$P_{cr} = 12.9\frac{EI}{L^2} \tag{4.3.11}$$

is obtained. This load is the same as before using the differential equation approach.

### 4.3.2 Sway-Permitted Case

Referring to Fig. 4.8a, we see that the slope-deflection equations (3.8.1) and (3.8.2) for the swayed column are

$$M_{AB} = \frac{EI_c}{L_c}\left[s_{iic}\theta_A + s_{ijc}\theta_B - (s_{iic} + s_{ijc})\frac{\Delta}{L_c}\right] = 0 \tag{4.3.12}$$

$$M_{BA} = \frac{EI_c}{L_c}\left[s_{ijc}\theta_A + s_{iic}\theta_B - (s_{iic} + s_{ijc})\frac{\Delta}{L_c}\right] \tag{4.3.13}$$

Solving Eq. (4.3.12) for $\theta_A$ and substituting $\theta_A$ into Eq. (4.2.13), we obtain

$$M_{BA} = \frac{EI_c}{L_c}\left[\left(s_{iic} - \frac{s_{ijc}^2}{s_{iic}}\right)\theta_B - \left(s_{iic} - \frac{s_{ijc}^2}{s_{iic}}\right)\frac{\Delta}{L_c}\right] \tag{4.3.14}$$

Since the beam is bent in double curvature, we use the slope-deflection equation (3.8.17) for the beam

$$M_{BC} = \frac{EI_b}{L_b}(s_{iib} + s_{ijb})\theta_B \tag{4.3.15}$$

Because there is no axial force in the beam, we set $s_{iib} = 4$ and $s_{ijb} = 2$, or

$$M_{BC} = \frac{6EI_b}{L_b}\theta_B \tag{4.3.16}$$

From joint equilibrium (Fig. 4.8b), we know

$$M_{BA} + M_{BC} = 0 \tag{4.3.17}$$

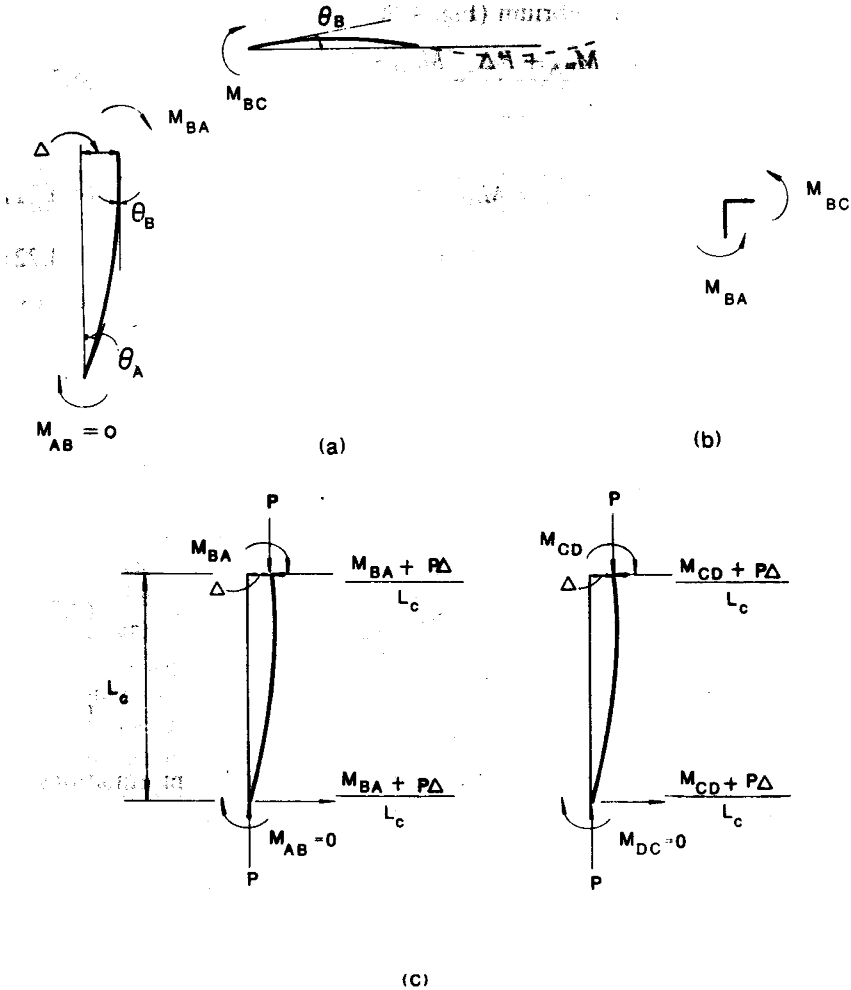



**FIGURE 4.8** Slope-deflection equation approach for $P_{cr}$ of sway buckling of simple portal frame

Using Eqs. (4.3.14) and (4.3.16), the joint equilibrium condition (4.3.17) becomes

$$\frac{EI_c}{L_c}\left[\left(s_{iic} - \frac{s_{ijc}^2}{s_{iic}}\right)\theta_B - \left(s_{iic} - \frac{s_{ijc}^2}{s_{iic}}\right)\frac{\Delta}{L_c}\right] + \frac{6EI_b}{L_b}\theta_B = 0 \tag{4.3.18}$$

or

$$\left(s_{iic} - \frac{s_{ijc}^2}{s_{iic}} + 6\frac{I_b L_c}{I_c L_b}\right)\theta_B - \left(s_{iic} - \frac{s_{ijc}^2}{s_{iic}}\right)\frac{\Delta}{L_c} = 0 \tag{4.3.19}$$

From story shear equilibrium (Fig. 4.8c), we have

$$\frac{M_{AB}+M_{BA}+P\Delta}{L_c}+\frac{M_{CD}+M_{DC}+P\Delta}{L_c}=0 \tag{4.3.20}$$

Realizing that

$$M_{AB}=M_{DC}=0 \quad \text{(hinged)} \tag{4.3.21}$$

and

$$M_{CD}=M_{BA} \quad \text{(antisymmetry)} \tag{4.3.22}$$

we can write the story-shear equilibrium equation (4.3.20) as

$$\frac{2M_{BA}+2P\Delta}{L_c}=0 \tag{4.3.23}$$

or

$$\frac{M_{BA}+P\Delta}{L_c}=0 \tag{4.3.24}$$

Using Eq. (4.3.14) for $M_{BA}$ in Eq. (4.3.24), we can write

$$\frac{EI_c}{L_c^2}\left[\left(s_{iic}-\frac{s_{ijc}^2}{s_{iic}}\right)\theta_B-\left(s_{iic}-\frac{s_{ijc}^2}{s_{iic}}\right)\frac{\Delta}{L_c}\right]+\frac{P\Delta}{L_c}=0 \tag{4.3.25}$$

or

$$\left(s_{iic}-\frac{s_{ijc}^2}{s_{iic}}\right)\theta_B-\left(s_{iic}-\frac{s_{ijc}^2}{s_{iic}}-k_c^2L_c^2\right)\frac{\Delta}{L_c}=0 \tag{4.3.26}$$

Equations (4.3.19) and (4.3.26) are the two equilibrium equations of the frame, they can be written in matrix form

$$\begin{bmatrix} S+6\dfrac{I_bL_c}{I_cL_b} & -S \\ S & -S+(k_cL_c)^2 \end{bmatrix}\begin{pmatrix}\theta_B \\ \dfrac{\Delta}{L_c}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \tag{4.3.27}$$

where

$$S=s_{iic}-\frac{s_{ijc}^2}{s_{iic}}$$

Note that the coefficient matrix in Eq. (4.3.27) can be made symmetric by multiplying Eq. (4.3.26) by minus one. If we do this, and also let $I_b=I_c=I$ and $L_b=L_c=L$, Eq. (4.3.27) becomes

$$\begin{bmatrix} S+6 & -S \\ -S & S-(kL)^2 \end{bmatrix}\begin{pmatrix}\theta_B \\ \dfrac{\Delta}{L}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \tag{4.3.28}$$

At bifurcation, both $\theta_B$ and $\Delta$ increase without bound. For Eq.

(4.3.28) to be valid, we must set

$$\det \begin{vmatrix} S+6 & -S \\ -S & S-(kL)^2 \end{vmatrix} = 0 \tag{4.3.29}$$

Equation (4.3.29) is the characteristic equation of the frame. By trial and error and by using Table 3.7, the value $kL$ can be found to be

$$kL = (\sqrt{P/EI})L = 1.35 \tag{4.3.30}$$

from which the critical load can be solved

$$P_{cr} = 1.82\frac{EI}{L^2} \tag{4.3.31}$$

Note the correspondence of Eq. (4.3.31) obtained using the slope-deflection method with Eq. (4.2.57) obtained previously using the differential equation method.

The slope-deflection equation method, as in the differential equation method, can in theory, be extended to evaluate $P_{cr}$ for all types of frames. The resulting coefficient matrix obtained by enforcing joint (and story-shear) equilibrium will be an $n \times n$ matrix in which $n$ is the number of independent degrees of freedom of the frame. However, if $n$ is large, it is cumbersome to obtain a solution. In the next section, the slope-deflection equation method will be generalized; the resulting formulation we will see is called the *matrix stiffness* method.[2,3] This procedure to obtain solutions for large frames can be greatly enhanced by the use of computers.

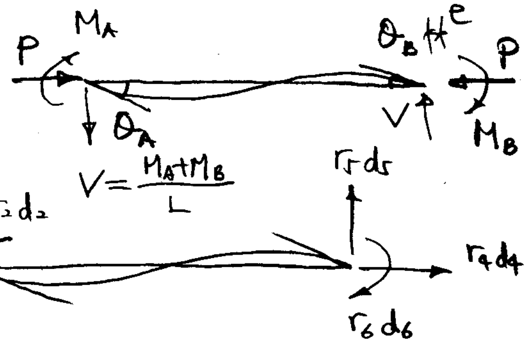
## 4.4 ELASTIC CRITICAL LOADS BY MATRIX STIFFNESS METHOD

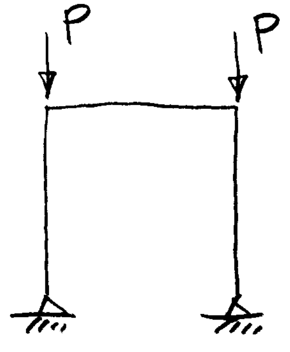
In the matrix stiffness method, the element stiffness matrix that relates the element end forces to end displacements is first formulated for each and every member of the frame. These element stiffness matrices are then assembled into the structure stiffness matrix that relates the structure nodal force to the structure nodal displacements. At bifurcation, the stiffness of the structure vanishes. Therefore, by setting the determinant of the structure stiffness matrix to zero, the critical load of the frame can be obtained.

### 4.4.1 Element Stiffness Formulation

We shall begin our discussion of the matrix stiffness method by developing the element stiffness matrix from the slope-deflection equation. Figure 4.9a shows the sign convention for the positive directions of

## Matrix Stiffness Method, $P_{cr}$




Eq (4.4.16), (4.4.23)

Taylor series ; (4.4.25 − 26)

1st two terms

2nd-Order Stiffness Matrix

$k = k_o + k_g$

$k_o$ = first-order linear stiffness matrix

Eq. (4.4.27)

$k_g$ = geometric stiffness matrix

Eq. (4.4.27)

$\det |k| = 0$. for bifurcation

$P_{cr} = \frac{1.03 EI}{L^2}$ vs. $\frac{1.02 EI}{L^2}$

round-off error

P 254 ~ 266

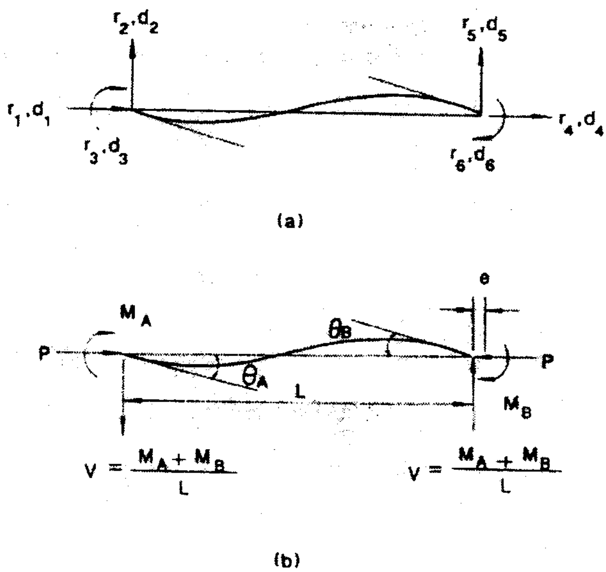



**FIGURE 4.9** Element end forces and displacements notations

element end forces and end displacements of a frame member. The end forces and end displacements used in the slope-deflection equation are shown in Fig. 4.9b. By comparing the two figures, we can easily express the following equilibrium and kinematic relationships.

### *Equilibrium*

$$r_1 = P \tag{4.4.1}$$

$$r_2 = -V = -\frac{M_A + M_B}{L} \tag{4.4.2}$$

$$r_3 = M_A \tag{4.4.3}$$

$$r_4 = -P \tag{4.4.4}$$

$$r_5 = \frac{M_A + M_B}{L} \tag{4.4.5}$$

$$r_6 = M_B \tag{4.4.6}$$

### *Kinematic*

(변형관계식 (적합방정식))

Shortening

$$e = -(d_4 - d_1) \tag{4.4.7}$$

$$\theta_A = d_3 + \left(\frac{d_5 - d_2}{L}\right) \tag{4.4.8}$$

$$\theta_B = d_6 + \left(\frac{d_5 - d_2}{L}\right) \tag{4.4.9}$$

Equations (4.4.1) to (4.4.6) can be written in matrix form as

$$\begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\dfrac{1}{L} & -\dfrac{1}{L} \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & \dfrac{1}{L} & \dfrac{1}{L} \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} P \\ M_A \\ M_B \end{pmatrix} \tag{4.4.10}$$

Similarly, Eqs. (4.7) to (4.4.9) can be written in matrix form as

$$\begin{pmatrix} e \\ \theta_A \\ \theta_B \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -\dfrac{1}{L} & 1 & 0 & \dfrac{1}{L} & 0 \\ 0 & -\dfrac{1}{L} & 0 & 0 & \dfrac{1}{L} & 1 \end{bmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{pmatrix} \tag{4.4.11}$$

Equation (4.4.10) and Eq. (4.4.11) can be related by recognizing that

$$P = \frac{EA}{L} e \tag{4.4.12}$$

$$M_A = \frac{EI}{L}(s_{ii}\theta_A + s_{ij}\theta_B) \tag{4.4.13}$$

$$M_B = \frac{EI}{L}(s_{ij}\theta_A + s_{ii}\theta_B) \tag{4.4.14}$$

Equation (4.4.12) relates the axial force $P$ to the axial displacement $e$ of the member, Eqs. (4.4.13) and (4.4.14) are the slope-deflection equations of the member, and $s_{ii}$, $s_{ij}$ are the stability functions. In writing Eq. (4.4.12), it is tacitly assumed that the effect of member shortening due to the bending curvature is negligible. This assumption is satisfactory for most practical purposes.

Putting Eqs. (4.4.12) to (4.4.14) in matrix form, we have

$$\begin{pmatrix} P \\ M_A \\ M_B \end{pmatrix} = \frac{EI}{L} \begin{bmatrix} \dfrac{A}{I} & 0 & 0 \\ 0 & s_{ii} & s_{ij} \\ 0 & s_{ij} & s_{ii} \end{bmatrix} \begin{pmatrix} e \\ \theta_A \\ \theta_B \end{pmatrix} \tag{4.4.15}$$

Substituting Eq. (4.4.15) into Eq. (4.4.10), and then substituting Eq. (4.4.11) into the resulting equation, we can relate the element end forces ($r_1$ to $r_6$) with the element end displacements ($d_1$ to $d_6$) as

$$\begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{pmatrix}_{ns} = \frac{EI}{L} \begin{bmatrix} \frac{A}{I} & 0 & 0 & -\frac{A}{I} & 0 & 0 \\ & \frac{2(s_{ii}+s_{ij})}{L^2} & -\frac{(s_{ii}+s_{ij})}{L} & 0 & \frac{-2(s_{ii}+s_{ij})}{L^2} & \frac{-(s_{ii}+s_{ij})}{L} \\ & & s_{ii} & 0 & \frac{s_{ii}+s_{ij}}{L} & s_{ij} \\ & \text{sym.} & & \frac{A}{I} & 0 & 0 \\ & & & & \frac{2(s_{ii}+s_{ij})}{L^2} & \frac{(s_{ii}+s_{ij})}{L} \\ & & & & & s_{ii} \end{bmatrix}_{ns} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{pmatrix} \tag{4.4.16}$$

" Stiffness metrix "

no sidesway

Symbolically, Eq. (4.4.16) can be written as

$$\mathbf{r}_{ns} = \mathbf{k}_{ns}\mathbf{d} \tag{4.4.17}$$

ㄷ 비변형

where the subscript ns is used here to indicate that there is no sidesway in the member. If the member is permitted to sway as shown in Fig. 4.10, an additional shear force equal to $P\,\Delta/L$ will be induced in the member due to the swaying of the member by an amount $\Delta$ given by

$$\Delta = d_2 - d_5 \tag{4.4.18}$$

We can relate this additional shear force due to member sway to the member end displacement as

$$\begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{pmatrix}_{s} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & -\frac{P}{L} & 0 & 0 & \frac{P}{L} & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & \text{sym.} & & & -\frac{P}{L} & 0 \\ & & & & & 0 \end{bmatrix}_{s} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{pmatrix} \tag{4.4.19}$$

sidesway 에 기한 추가분 전단력

sidesway

or symbolically

$$\mathbf{r}_s = \mathbf{k}_s\mathbf{d} \tag{4.4.20}$$

횡변위 방향의 Sidesway 에 의한 stiffness metrix

where the subscript s is used to indicate the quantities due to sidesway of the member.

By combining Eq. (4.4.17) and Eq. (4.4.20), we obtain the general beam-column element force-displacement relationship as

$$\mathbf{r} = \mathbf{k}\mathbf{d} \tag{4.4.21}$$

where

$$\mathbf{r} = \underset{k_{ns}d}{\mathbf{r}_{ns}} + \underset{k_s \cdot d}{\mathbf{r}_s} \tag{4.4.22a}$$

$$\mathbf{k} = \mathbf{k}_{ns} + \mathbf{k}_s \quad \text{: stiffness metrix} \tag{4.4.22b}$$

P항 포함

$$\mathbf{k} = \frac{EI}{L}\begin{bmatrix} \frac{A}{I} & 0 & 0 & -\frac{A}{I} & 0 & 0 \\ & \frac{2(s_{ii}+s_{ij})-(kL)^2}{L^2} & -\frac{(s_{ii}+s_{ij})}{L} & 0 & \frac{-2(s_{ii}+s_{ij})+(kL)^2}{L^2} & \frac{-(s_{ii}+s_{ij})}{L} \\ & & s_{ii} & 0 & \frac{s_{ii}+s_{ij}}{L} & s_{ij} \\ & \text{sym.} & & \frac{A}{I} & 0 & 0 \\ & & & & \frac{2(s_{ii}+s_{ij})-(kL)^2}{L^2} & \frac{(s_{ii}+s_{ij})}{L} \\ & & & & & s_{ii} \end{bmatrix} \tag{4.4.23}$$

Substituting the expressions for the stability functions $(s_{ii}, s_{ij})$ in Eq. (4.4.23) and simplifying, we obtain

$$\mathbf{k} = \frac{EI}{L}\begin{bmatrix} \frac{A}{I} & 0 & 0 & -\frac{A}{I} & 0 & 0 \\ & \frac{12}{L^2}\phi_1 & \frac{-6}{L}\phi_2 & 0 & \frac{-12}{L^2}\phi_1 & \frac{-6}{L}\phi_2 \\ & & 4\phi_3 & 0 & \frac{6}{L}\phi_2 & 2\phi_4 \\ & & & \frac{A}{I} & 0 & 0 \\ & \text{sym.} & & & \frac{12}{L^2}\phi_1 & \frac{6}{L}\phi_2 \\ & & & & & 4\phi_3 \end{bmatrix} \tag{4.4.24}$$

$\phi_1, \phi_2 = 0$ 일때 선형 stiffness metrix

The expressions for $\phi_1$, $\phi_2$, $\phi_3$, and $\phi_4$ are given in Table 4.1. Note that as $P$ approaches zero, the functions $\phi_1$, $\phi_2$, $\phi_3$, and $\phi_4$ become

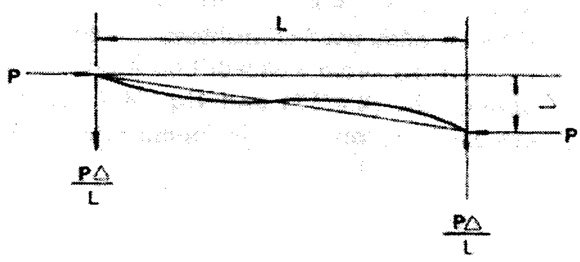



FIGURE 4.10 Additional shear due to swaying of the member

indefinite. However, by using the L'Hospital's rule, it can be shown that these functions will approach unity and Eq. (4.4.24) reduces to the first-order (linear) element stiffness matrix for a frame member.

Also shown in Table 4.1 are the $\phi_i$ functions expressed in the form of a power series by using the following series expansion for the trigonometric functions:

For compression

radian 값

$$\sin kL = kL - \frac{(kL)^3}{6} + \frac{(kL)^5}{120} + \cdots \qquad (4.4.25a)$$

$$\cos kL = 1 - \frac{kL}{2} + \frac{(kL)^4}{24} + \cdots \qquad (4.4.25b)$$

For tension

$$\sinh kL = kL + \frac{(kL)^3}{6} + \frac{(kL)^5}{120} + \cdots \qquad (4.4.26a)$$

$$\cosh kL = 1 + \frac{kL}{2} + \frac{(kL)^4}{24} + \cdots \qquad (4.4.26b)$$

It has been shown[4] that these power series expressions are convenient and efficient to use in a computer-aided analysis because no numerical difficulties will arise even if the axial force $P$ is small. In addition, the expressions in the series are the same regardless of whether $P$ is tensile or compressive. For most cases, the series will converge to a high degree of accuracy if $n = 10$ is used.

If the axial force in the member is small, Eq. (4.4.24) can be simplified by using a Taylor series expansion for the $\phi_i$'s. If we retain only the first two terms in the Taylor series, it can be shown that the resulting stiffness

P가 0이면 ∅값은 0이 되기때문에 stable한 값을 얻기위해서는
power series는 사용할수밖에 없다.



**Table 4.1** Expressions for $\phi_1$, $\phi_2$, $\phi_3$, and $\phi_4$

| $\phi$ | $P$ | | |
|---|---|---|---|
| | Compressive | Zero | Tensile |
| $\phi_1$ | $\dfrac{(kL)^3 \sin kL}{12\phi_c}$ | 1 | $\dfrac{(kL)^3 \sinh kL}{12\phi_t}$ |
| $\phi_2$ | $\dfrac{(kL)^2(1-\cos kL)}{6\phi_c}$ | 1 | $\dfrac{(kL)^2(\cosh kL-1)}{6\phi_t}$ |
| $\phi_3$ | $\dfrac{(kL)(\sin kL - kL\cos kL)}{4\phi_c}$ | 1 | $\dfrac{(kL)(kL\cosh kL-\sinh kL)}{4\phi_t}$ |
| $\phi_4$ | $\dfrac{(kL)(kL-\sin kL)}{2\phi_c}$ | 1 | $\dfrac{(kL)(\sinh kL - kL)}{2\phi_t}$ |

where

$$\phi_c = 2 - 2\cos kL - kL\sin kL \qquad \phi_t = 2 - 2\cosh kL + kL\sinh kL$$

Alternatively, the $\phi_i$ functions can be expressed in the form of power series, as in reference 4:

$$\phi_1 = \frac{1 + \sum_{n=1}^{\infty} \frac{1}{(2n+1)!}[\mp(kL)^2]^n}{12\phi}$$

: power series로 expansion

$$\phi_2 = \frac{\frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{(2n+2)!}[\mp(kL)^2]^n}{6\phi}$$

$$\phi_3 = \frac{\frac{1}{3} + \sum_{n=1}^{\infty} \frac{2(n+1)}{(2n+3)!}[\mp(kL)^2]^n}{4\phi}$$

$$\phi_4 = \frac{\frac{1}{6} + \sum_{n=1}^{\infty} \frac{1}{(2n+3)!}[\mp(kL)^2]^n}{2\phi}$$

where

$$\phi = \frac{1}{12} + \sum_{n=1}^{\infty} \frac{2(n+1)}{(2n+4)!}[\mp(kL)^2]^n$$

Use the minus sign if the axial force is compressive.
Use the plus sign if the axial force is tensile.

power series의 처음의 두개까지만 쓰면

matrix that is valid for small axial force is given by

: 일반적인 program 는 이 두개만 고려함.

$$
\mathbf{k}=\frac{EI}{L}\begin{bmatrix}
\frac{A}{I} & 0 & 0 & -\frac{A}{I} & 0 & 0\\
 & \frac{12}{L^2} & \frac{-6}{L} & 0 & \frac{-12}{L^2} & \frac{-6}{L}\\
 & & 4 & 0 & \frac{6}{L} & 2\\
 & & & \frac{A}{I} & 0 & 0\\
 & & & & \frac{12}{L^2} & \frac{6}{L}\\
 & & & & & 4
\end{bmatrix}
\mp\frac{P}{L}\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0\\
 & \frac{6}{5} & \frac{-L}{10} & 0 & \frac{-6}{5} & \frac{-L}{10}\\
 & & \frac{2L^2}{15} & 0 & \frac{L}{10} & \frac{-L^2}{30}\\
 & & & 0 & 0 & 0\\
 & & & & \frac{6}{5} & \frac{L}{10}\\
 & & & & & \frac{2L^2}{15}
\end{bmatrix}
\tag{4.4.27}
$$

$K_0$ $K_G$

P가 크지 않을때는 축 P ~ 커지면 정확하게 가 떨어짐.

P가 큰 경우는 원래 matrix 사용

powerseries 2개만

) 2차항 안함

선형 matrix

비선형 혹은 2차항 matrix

in which the negative sign preceding the second matrix corresponds to a compressive axial force and the positive sign corresponds to a tensile axial force.

Symbolically, Eq. (4.4.27) can be written as

$$\mathbf{k}=\mathbf{k}_0+\mathbf{k}_G \tag{4.4.28}$$

where $\mathbf{k}_0$ is the first-order (linear) elastic stiffness matrix and $\mathbf{k}_G$ is the geometric stiffness matrix (sometimes referred to as the initial stress stiffness matrix), which accounts for the effect of the axial force $P$ on the bending stiffness of the member.

The following example will be used to demonstrate the procedure of using the stiffness matrix method to obtain the critical load of frames.

### 4.4.2 Sway Buckling of a Pinned-Base Portal Frame

The matrix stiffness method is applied here to determine the critical load $P_{cr}$ for the frame shown in Fig. 4.5a. Because of symmetry, we consider only one half of the structure in the analysis. This is shown in Fig. 4.11a together with the structural nodal forces and displacements. To reduce the number of degrees of freedom of the structure, we assume that all members are inextensible (i.e., the change in length due to axial force is neglected). As a result, only four degrees of freedom, are labeled: three rotational degrees of freedom, $D_1$, $D_2$, and $D_3$, and one translational degree of freedom, $D_4$. The corresponding structural nodal forces, $R_1, \ldots, R_4$, are also shown in Fig. 4.11a. The directions of these

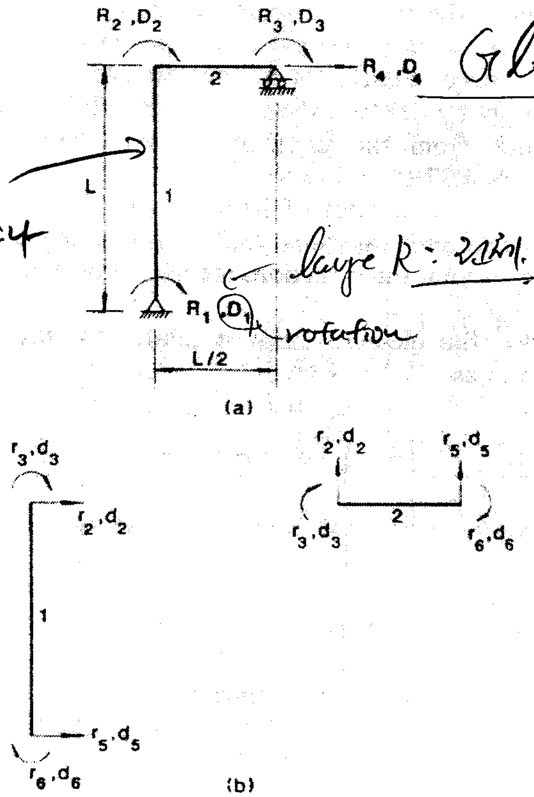



**FIGURE 4.11** Structure and member forces and displacements notations

rotations, translations, and forces are shown in their positive sense in the figure.

Because of the assumption of inextensional behavior, the axial force-axial displacement relationship expressed in Eq. (4.4.12) is not valid anymore. As a consequence, the $6 \times 6$ element stiffness matrix relating the element end forces to the element end displacements will be reduced to a $4 \times 4$ matrix as

$$\mathbf{k} = \frac{EI}{L}\begin{bmatrix} \frac{12}{L^2} & -\frac{6}{L} & -\frac{12}{L^2} & -\frac{6}{L} \\ & 4 & \frac{6}{L} & 2 \\ \text{sym.} & & \frac{12}{L^2} & \frac{6}{L} \\ & & & 4 \end{bmatrix} \mp \frac{P}{L}\begin{bmatrix} \frac{6}{5} & -\frac{L}{10} & -\frac{6}{5} & -\frac{L}{10} \\ & \frac{2L^2}{15} & \frac{L}{10} & -\frac{L^2}{30} \\ \text{sym.} & & \frac{6}{5} & \frac{L}{10} \\ & & & \frac{2L^2}{15} \end{bmatrix} \tag{4.4.29}$$

This stiffness matrix relates the four end forces ($r_2$, $r_3$, $r_5$, and $r_6$) to the four end displacements ($d_2$, $d_3$, $d_5$, and $d_6$) of an inextensible member. Note that the element stiffness matrix for an inextensible member [Eq. (4.4.29)] is obtained simply by deleting the first and fourth rows and the first and fourth columns from the element stiffness matrix for an extensible member [Eq. (4.4.27)].

Figure 4.11b shows the four degrees of freedom ($d_2$, $d_3$, $d_5$, and $d_6$) and the corresponding end forces ($r_2$, $r_3$, $r_5$, and $r_6$) associated with each member of the structure. Again, the directions are shown in their positive sense in the figure.

By using Eq. (4.4.29), the element stiffness matrix for the column (element 1) can be written as

$$\mathbf{k}_1 = \frac{EI}{L}\begin{bmatrix} \frac{12}{L^2} & \frac{-6}{L} & \frac{-12}{L^2} & \frac{-6}{L} \\ & 4 & \frac{6}{L} & 2 \\ & & \frac{12}{L^2} & \frac{6}{L} \\ \text{sym.} & & & 4 \end{bmatrix} - \frac{P}{L}\begin{bmatrix} \frac{6}{5} & \frac{-L}{10} & \frac{-6}{5} & \frac{-L}{10} \\ & \frac{2L^2}{15} & \frac{L}{10} & \frac{-L^2}{30} \\ & & \frac{6}{5} & \frac{L}{10} \\ \text{sym.} & & & \frac{2L^2}{15} \end{bmatrix} \tag{4.4.30}$$

and the element stiffness matrix for beam with $P = 0$ and $L/2$ for $L$ (element 2) can be written as

$$\mathbf{k}_2 = 2\frac{EI}{L}\begin{bmatrix} \frac{48}{L^2} & \frac{-12}{L} & \frac{-48}{L^2} & \frac{-12}{L} \\ & 4 & \frac{12}{L} & 2 \\ & & \frac{48}{L^2} & \frac{12}{L} \\ \text{sym.} & & & 4 \end{bmatrix} \tag{4.4.31}$$

The structure stiffness matrix can be obtained by assembling these element stiffness matrices. The process of assemblage is described in detail in most matrix structural analysis textbooks.[5–7] So we will discuss it only very briefly here.

For each element, the element end displacements are first related to the structure nodal displacements by consideration of joint compatibility. It can easily be seen from Fig. 4.11 that for element 1, this kinematic relationship is

Local 좌표계 Global 좌표

$$\begin{Bmatrix} d_2 \\ d_3 \\ d_5 \\ d_6 \end{Bmatrix}_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}_1 \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{Bmatrix} \tag{4.4.32}$$

For element 2, the kinematic relationship is

$$\begin{Bmatrix} d_2 \\ d_3 \\ d_5 \\ d_6 \end{Bmatrix}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}_2 \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{Bmatrix} \tag{4.4.33}$$

Symbolically, Eqs. (4.4.32) and (4.4.33) can be written respectively as

1번 부재

$$\mathbf{d}_1 = \mathbf{T}_1\mathbf{D} \tag{4.4.34}$$

transformal matrix

$$\mathbf{d}_2 = \mathbf{T}_2\mathbf{D} \tag{4.4.35}$$

부재 2번

On the other hand, the portion of the structure nodal forces resisted by element 1 is

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix}_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} r_2 \\ r_3 \\ r_5 \\ r_6 \end{Bmatrix}_1 \tag{4.4.36}$$

transpose관계

and the portion of the structural nodal force resisted by element 2 is

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} r_2 \\ r_3 \\ r_5 \\ r_6 \end{Bmatrix}_2 \tag{4.4.37}$$

By comparing Eq. (4.4.36) with Eq. (4.4.32) and Eq. (4.4.37) with Eq. (4.4.33), it can be seen that the matrix relating the structure nodal forces $R$'s to the element end forces $r$'s is the transposition of the matrix relating the element end displacements $d$'s to the structure nodal displacements $D$'s. This observation is not a coincidence, but represents a theory in structural analysis known as the *contragradient* law.[7]

In view of the above observation, Eqs. (4.4.36) and (4.4.37) can be written symbolically as

$$\mathbf{R}_1 = \mathbf{T}_1^T \mathbf{r}_1 \tag{4.4.38}$$

$$\mathbf{R}_2 = \mathbf{T}_2^T \mathbf{r}_2 \tag{4.4.39}$$

From consideration of joint equilibrium, we can write

$$\mathbf{R} = \mathbf{R}_1 + \mathbf{R}_2 \tag{4.4.40}$$

Substituting the member equilibrium relationships Eqs. (4.4.38) and (4.4.39) into Eq. (4.4.40) gives

$$\mathbf{R} = \mathbf{T}_1^T \mathbf{r}_1 + \mathbf{T}_2^T \mathbf{r}_2 \tag{4.4.41}$$

Since, from Eq. (4.4.21) the element force-displacement relationship for elements 1 and 2 can be written, respectively, as

$$\mathbf{r}_1 = \mathbf{k}_1 \mathbf{d}_1 \tag{4.4.42}$$

and

$$\mathbf{r}_2 = \mathbf{k}_2 \mathbf{d}_2 \tag{4.4.43}$$

we can write Eq. (4.4.41) as

$$\mathbf{R} = \mathbf{T}_1^T \mathbf{k}_1 \mathbf{d}_1 + \mathbf{T}_2^T \mathbf{k}_2 \mathbf{d}_2 \tag{4.4.44}$$

Now, using the member kinematic relationships, Eqs. (4.4.34) and (4.4.35), we can write Eq. (4.4.44) as

$$\begin{aligned} \mathbf{R} &= \mathbf{T}_1^T \mathbf{k}_1 \mathbf{T}_1 \mathbf{D} + \mathbf{T}_2^T \mathbf{k}_2 \mathbf{T}_2 \mathbf{D} \\ &= (\mathbf{T}_1^T \mathbf{k}_1 \mathbf{T}_1 + \mathbf{T}_2^T \mathbf{k}_2 \mathbf{T}_2)\mathbf{D} \end{aligned} \tag{4.4.45}$$

or

$$\mathbf{R} = \mathbf{K}\mathbf{D} \tag{4.4.46}$$

where

$$\boxed{\mathbf{K} = \mathbf{T}_1^T \mathbf{k}_1 \mathbf{T}_1 + \mathbf{T}_2^T \mathbf{k}_2 \mathbf{T}_2} \quad \text{Global stiffness mat} \tag{4.4.47}$$

is the structure stiffness matrix.

The process shown above is referred to as assemblage and it involves the process of transforming and putting together element stiffness matrices to form the structure stiffness matrix. In general, if these are $n$ elements in the structure, the structure stiffness matrix can be obtained as

$$\boxed{\mathbf{K} = \sum_{i=1}^{n} \mathbf{T}_i^T \mathbf{k}_i \mathbf{T}_i} \tag{4.4.48}$$

Now, referring back to the example problem, upon substituting the matrices $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{k}_1$, $\mathbf{k}_2$ into the structure stiffness matrix Eq. (4.4.47) and carrying out the matrix products, we see that the structure stiffness matrix can be written as

$$\mathbf{K}=\frac{EI}{L}\begin{bmatrix} 4 & 2 & 0 & \frac{-6}{L} \\ & 12 & 4 & \frac{-6}{L} \\ & & 8 & 0 \\ \text{sym.} & & & \frac{12}{L^2} \end{bmatrix} - \frac{P}{L}\begin{bmatrix} \frac{2L^2}{15} & \frac{-L^2}{30} & 0 & \frac{-L}{10} \\ & \frac{2L^2}{15} & 0 & \frac{-L}{10} \\ & & 0 & 0 \\ \text{sym.} & & & \frac{6}{5} \end{bmatrix} \tag{4.4.49}$$

Denoting

$$\lambda=\frac{PL^2}{30EI}=\frac{(kL)^2}{30} \tag{4.4.50}$$

Eq. (4.4.49) can be written as

$$\mathbf{K}=\frac{EI}{L}\begin{bmatrix} 4-4\lambda & 2+\lambda & 0 & \frac{-6+3\lambda}{L} \\ & 12-4\lambda & 4 & \frac{-6+3\lambda}{L} \\ & & 8 & 0 \\ \text{sym.} & & & \frac{12-36\lambda}{L^2} \end{bmatrix} \tag{4.4.51}$$

At bifurcation, the determinant of the stiffness matrix must vanish. Thus, by setting

$$\det|\mathbf{K}|=0 \tag{4.4.52}$$

we obtain a polynomial in $\lambda$. The smallest root satisfying this equation is $\lambda=0.061$, and from Eq. (4.4.50)

$$P_{cr}=30\lambda\frac{EI}{L^2}=1.83\frac{EI}{L^2} \tag{4.4.53}$$

The slight discrepancy of Eq. (4.4.53) compared to the value obtained previously by the differential equation method or the slope-deflection equation method is due to the round-off error, and this error was introduced earlier as a result of the approximation from Eq. (4.4.24) to Eq. (4.4.27).

At first glance, it seems that there is much more work involved in the stiffness matrix approach than that of the differential equation or the slope-deflection equation approaches. However, it should be noted that the steps shown above can easily be programmed in a digital computer, and so $P_{cr}$ can be obtained quite conveniently for any type of frame.

## 4.5 SECOND-ORDER ELASTIC ANALYSIS

In the preceding sections, we determined the load that corresponds to a state of bifurcation of equilibrium of a perfect frame by an eigenvalue analysis. In an eigenvalue analysis, the system is assumed to be perfect. There will be no lateral deflections in the members until the load reaches the critical load $P_{cr}$. At the critical load $P_{cr}$, the original configuration of the frame ceases to be stable and with a slight disturbance, the lateral deflections of the members begin to increase without bound as indicated by curve 1 in Fig. 4.2. However, if the system is not perfect, lateral deflections will occur as soon as the load is applied, as shown by curve 2 in Fig. 4.2. For an elastic frame, curve 2 will approach curve 1 asymptotically. To trace this curve, a complete load-deflection analysis of the frame is necessary. A second-order elastic analysis will generate just such load-deflection response of the frame.

In a second-order analysis, such secondary effects as the $P-\delta$ and $P-\Delta$ effects, which we discussed previously in Chapter 3, can be incorporated directly into the analysis procedure. As a result, the use of $P-\delta$ and $P-\Delta$ moment magnification factors (denoted as $B_1$ and $B_2$ in Chapter 3) are not necessary.

Because for a second-order analysis the equilibrium equations are formulated with respect to the deformed geometry of the structure, which is not known in advance and is constantly changing with the applied loads, it is necessary to employ an iterative technique to obtain solutions. In a numerical implementation, one of the most popular solution techniques is the incremental load approach. In this approach, the applied load is divided into increments and applied incrementally to the structure. The deformed configurations of the structure at the end of each cycle of calculation is used as the basis for the formulation of equilibrium equations for the next cycle. At a particular cycle of calculation, the structure is assumed to behave linearly. In effect, the nonlinear response of the structure as a result of geometric changes is approximated by a series of linear analyses, the geometry of the structure used in the analysis for a specific cycle is the deformed geometry of the structure corresponding to the previous cycle of calculation. Because of the linearization process, equilibrium may be violated and the external force may not always balance the internal force. This unbalanced force must be reapplied to the structure and the process repeated until equilibrium is satisfied.

For a second-order elastic frame analysis, the iteration process is

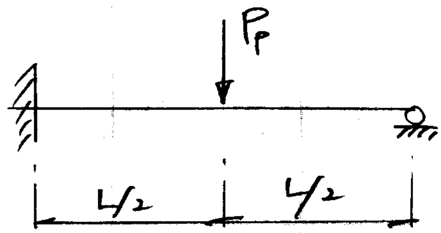
# Plastic Collapse Loads, $P_p$

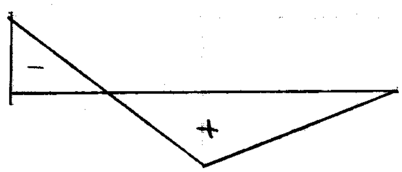
Methods

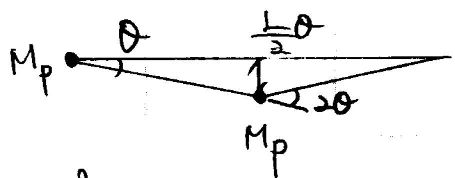
- Hinge-by-Hinge Method
- Mechanism Method * ; Failure Mechanism.

Chen 교수 책소개 ; 한학기 Course, 저서소개 (30), 강의

## Mechanism Method




Internal Work

$$W = M_p \cdot \theta + M_p \cdot 2\theta = 3M_p\theta$$

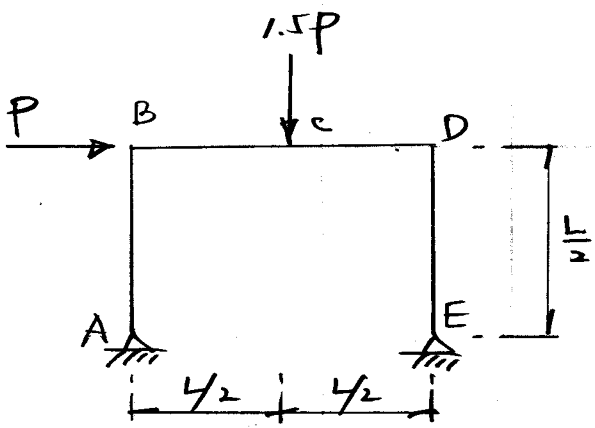
External Work

$$V = P_p \cdot \frac{1}{2} L\theta$$

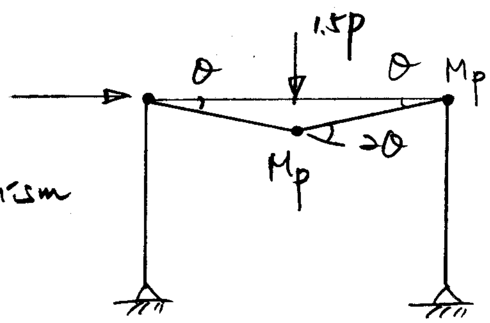
$$W = V$$

$$P_p = \frac{6M_p}{L}$$

## Example : Portal Frame




Beam Mechanism

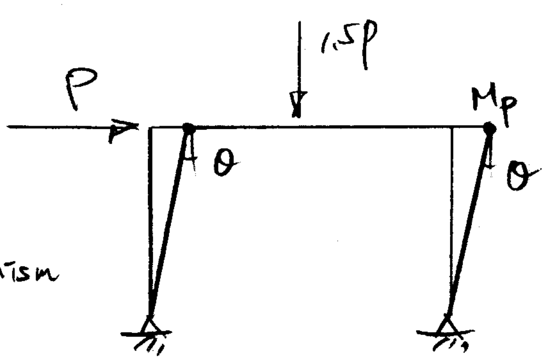



$U = V$

$4M_p\theta = 1.5P_1\frac{L}{2}\theta$

$P_1 = \frac{16}{3}\frac{M_p}{L}$ //

Sway Mechanism

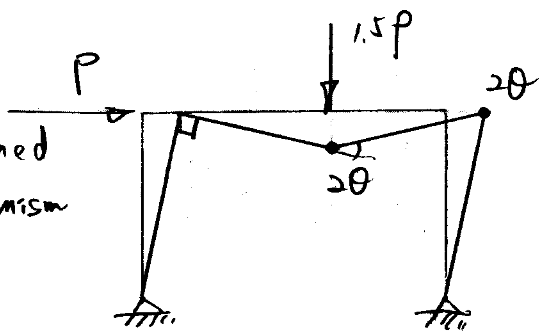



$U = V$

$2M_p\cancel{\theta} = P_2 \cdot \frac{L}{2}\cancel{\theta}$

$P_2 = \frac{4M_p}{L}$ //

Combined Mechanism




$U = V$

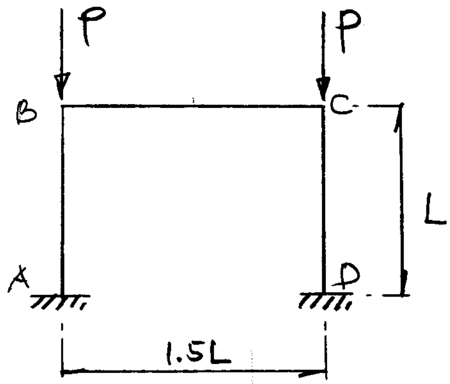
$4M_p\theta = P_3 \cdot \frac{L}{2}\theta + 1.5P_3 \cdot \frac{L}{2}\theta$

$P_3 = \frac{16M_p}{5L}$ //

$P_p = \mathrm{Min}\,(P_1 \cdots P_3) = \frac{16M_p}{5L}$ //

# Homework #9

1.



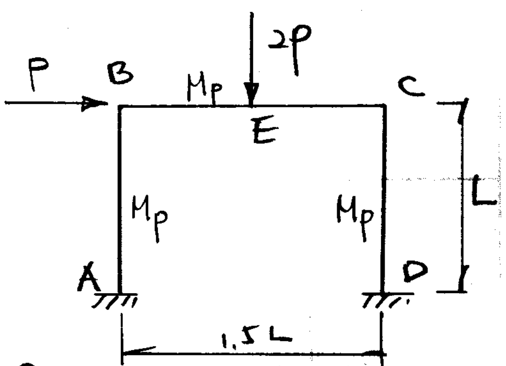

$EI = const.$

Slope – Deflection Eq를 사용하여 AB Member의
$P_{cr}$, K를 구하시오

Case 1 ; Braced

Case 2 ; Unbraced

2.




$EI = const$

$P_p$를 구하시오.

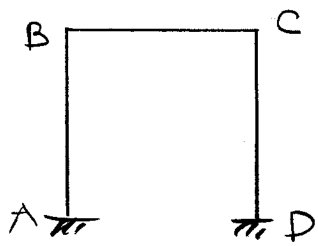
3. 문제 1번 및 2번의 결과를 활용하여 2번 문제에 주어진 Frame의 Approximate $P_f$를 구하시오. (Merchant-Rankine)

Today

Recall

$P_{cr}$ (Elastic Buckling Load)
$P_p$ (Plastic Collapse Load) $> P_{max}$ ($P_f$) : Merchant-Rankine

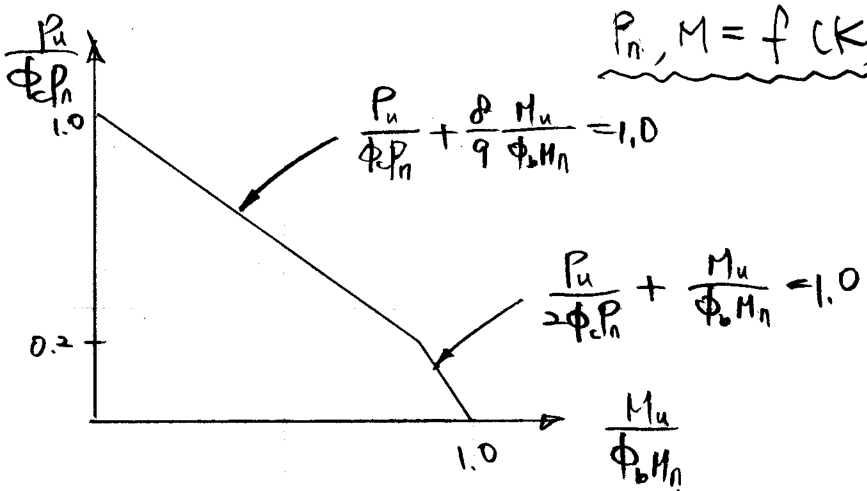
LRFD Procedure




AB Member Design : Axial force & Moment

→ Interaction Equation

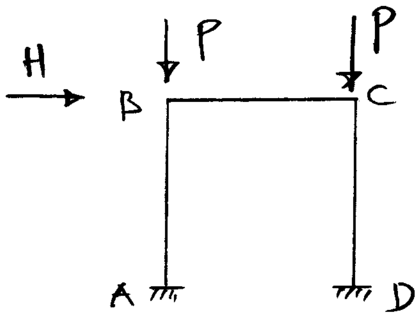
$P_n, M = f(K)$




K-factor

Design Example

## K- Factor




AB Column 의 K-factor ?

; Beam BC의 강성에 따라 변화

Stiffness Ratio $G = \frac{\Sigma(EI/L)_c}{\Sigma(EI/L)_b}$

$$G_A = \frac{(EI/L)_c}{\infty} = 0$$

$$G_B = \frac{(EI/L)_c}{(EI/L)_b} = 1$$

Alignment Chart (p.287 ; Unbraced Case
p.283 ; Braced Case)

$K = 1.15$

↓

Interaction Eq ; Member Strength Check